Advanced Macroeconomics 1

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September 10, 2019

Lecture 2:
Neoclassical Growth, foundations

Foundations of Neoclassical Growth

(ref: Acemoglu, ch 5)

- In the previous lecture, we assumed a constant saving rate
- In order to understand how different factors affect savings, we need to specify the preference ordering of households and derive saving decisions from these preferences.
- With well specified preferences we can also think about the desirability of different allocations.

Preliminaries

- A unit measure of infinitely-lived households (averages equal to aggregates)
- Households have well-defined preference orderings which can be represented by utility functions
- Household h has an instantaneous utility function (felicity function) given by

$$u^h(c^h(t)),$$

  where c^h is the household's consumption at t, and $u^h : \mathbb{R}_+ \to \mathbb{R}$
- Assumptions:
  1. No utility from consumption of other households
  2. time-separable and stationary preferences

- Households are assumed to discount the future exponentially
- In discrete time this implies that a household's utility can be represented as

$$U^h(c^h(0), c^h(1), c^h(2), ..., c^h(T)) = \sum_{t=0}^{T} (\beta^h)^t u(c^h(t)), \tag{1}$$

where $0 < \beta < 1$ and T could be infinite.

- This specification ensures time-consistent behaviour.
- A solution, $\{x_t\}_{t=0}^T$, to a dynamic problem is time-consistent if the following is true, when $\{x_t\}_{t=0}^T$ is a solution starting at t=0, $\{x_t\}_{t=t'}^T$ is a solution starting from from t'.

The representative household

- An economy admits a representative household when the demand side of the economy can be represented as if there was a single household making the consumption and saving decisions.
- A stronger notion, a normative representative household allows one also to do welfare analysis based on the preferences of the representative household.
- If all agents are identical, there naturally exists a normative representative household.
- When agents are heterogeneous, it depends on preferences whether a representative household exists.

The representative household: Gorman's aggregation theorem

- Consider an economy with $N < \infty$ commodities and a set $\mathcal{H}$ of households
  - When preferences of each household $h \in \mathcal{H}$ take the form

$$v^h(p, w) = a^h(p) + b(p)w^h, \tag{2}$$

    where p is a price vector and w^h is the HH's wealth,
  - and each household has positive demand for each commodity
  - then the preferences can be aggregated and represented by those of a representative household

$$v(p, w) = a(p) + b(p)w,$$

    where $a(p) \equiv \int_{h \in \mathcal{H}} a^h(p)dh$ and $w \equiv \int_{h \in \mathcal{H}} w^h dh$

- With ”Gorman preferences”, there is a linear relationship between income and consumption (for a given price)
- This condition has to hold if we wish to have a representative household (without imposing restrictions on the distribution of income)
- An economy admits a strong representative household if the redistribution of income (or endowments) does not affect the demand.
- The Gorman preferences also generally imply the existence of a normative representative household (see theorem 5.3 in Acemoglu’s book)

An example: CES preferences

- Each HH h
    - has preferences

$$U^h(x_1^h, ..., x_N^h) = [\sum_{j=1}^{N}(x_j^h - \xi_j^h)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}},$$

      where $0 < \sigma < \infty$ and $\xi_j^h \in [-\bar{\xi}, \bar{\xi}]$
    - faces prices $p = \{p_1, ..., p_N\}$
    - $\sum_{j=1}^{N} p_j \bar{\xi} < w^h$

- The household's problem

$$\max_{\{x_j\}} U_h(x_1, ..., x_N) - \lambda(w^h - \sum_{j=1}^{N} p_j x_j)$$

- Combining two FOCs for good i and j

$$\frac{x_j - \xi_j^h}{x_i - \xi_i^h} = \left(\frac{p_i}{p_j}\right)^\sigma$$

- Multiply by p_j and sum over js

$$\frac{w^h - \sum_{j=1}^{N} p_j \xi_j^h}{x_i - \xi_i^h} = p_i^\sigma \sum_{j=1}^{N} p_j^{1-\sigma}$$

- Thus

$$x_i - \xi_i^h = \frac{w^h - \sum_{j=1}^N p_j \xi_j^h}{p_i^\sigma \sum_{j=1}^N p_j^{1-\sigma}}$$

- Plug this into $U(x_1^h, ..., x_N)$ to get

$$\begin{aligned} v^h(p, w^h) &= \frac{w^h - \sum_{j=1}^N p_j \xi_j^h}{(\sum_{j=1}^N p_j^{1-\sigma})(\sum_{i=1}^N p_i^{1-\sigma})^{\frac{\sigma}{\sigma-1}}} \\ &= \frac{w^h - \sum_{j=1}^N p_j \xi_j^h}{\left(\sum_{j=1}^N p_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}} \end{aligned}$$

which satisfies the Gorman form.

Infinite planning horizon

- Most growth (and macro) models assume an infinite planning horizon
- Rationalizations:
  - perpetual youth model
  - intergenerational altruism
- The simplest form of the perpetual youth model:
  - individuals' utility follows the form stated in eq(1) with discount factor $\hat{\beta}$
  - in each period, there is a constant probability of death ν
  - Combining:

$$
\begin{aligned}
U(c(0), c(1), ...) = \quad & u(c(0)) + \hat{\beta}(1-\nu)u(c(1)) \\
& + \hat{\beta}^2(1-\nu)^2 u(c(2)) + ... \\
= \quad & \sum_{t=0}^{\infty} \beta^t u(c(t))
\end{aligned}
$$

Intergenerational altruism (the bequest motive)

- An individual
  - lives only for one period and has a single offspring (who also lives for one period and has a single offspring)
  - derives utility from his own consumption and from bequests
  - has a budget constraint of the form: $c(t) + b(t) \leq y(t)$
- Let the intergenerational discount factor be β and assume that the offspring will have an income w.
- The utility of an individual can be written as

$$u(c(t)) + \beta V(b(t) + w)$$

- The value of an individual can be written as

$$V(y(t)) = \max_{c(t)+b(t)\leq y(t)} \{u(c(t)) + \beta V(b(t) + w) \tag{3}$$

- As we see later, under some (relatively mild) conditions this is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

The representative firm

- When there are no externalities and all factors are priced competitively, we can represent the production side with a representative firm.
- We do not need to worry about income effects here.
- The next slide states this more formally (for proof, see Acemoglu p. 158)
- Notations:
  - $p \cdot y = \sum_{j=1}^{N} p_j y_j$
  - $\mathcal{F}$ is the set of firms in the economy
  - the aggregate production possibilities set of the economy is

$$Y \equiv \left\{ \sum_{f \in \mathcal{F}} y^f : y^f \in Y^f \, for \, each \, f \in \mathcal{F} \right\}$$

Theorem

Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a production possibilities set $Y^f \in \mathbb{R}^N$. Let $p \in \mathbb{R}^N_+$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a representative firm with production possibilities set $Y \subset \mathbb{R}^N$ and a set of profit maximizing net supplies $\hat{Y}(p)$, such that for any $p \in \mathbb{R}^N_+$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.

Problem formulation

- Consider a discrete time infinite-horizon economy that admits a representative household
- the HH's utility is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{4}$$

$0 < \beta < 1$

- In continuous time the problem takes the form

$$\int_0^{\infty} \exp(-\rho t) u(c(t)) dt, \tag{5}$$

where ρ is the discount rate.

- To see the connection between (4) and (5), think about continuous compound interest rate.

Welfare theorems

- We are interested in economic growth but there is a connection between competitive equilibrium and Pareto optima that turns out to be useful.
- Let's first define the structure of the economy.
- After that, we can define a competitive equilibrium and a Pareto efficient allocation for this economy.

Consumers

- Assume that we have a finite number of households but infinite (countable) number of commodities.
- Denote HH h's consumption bundle as $x^h = \{x_j^h\}_{j=0}^{\infty}$, where $h \in \mathcal{H}$ and $x^h \in X^h \subset \mathbb{R}_+^N$ (i.e. X^h is the consumption set for household h).
- An interpretation: an infinite number of days and for each day there is a finite number of goods. That is $\{\tilde{x}_{1,t}^h, ..., \tilde{x}_{N,t}^h\} \in \tilde{X}_t^h \subset \mathbb{R}_+^N$ for some $N \in \mathbb{N}$ and $x^h = \{\tilde{x}_t^h\}_{t=0}^{\infty}$
- Moreover, $\omega^h = \{\omega_j^h\}_{j=0}^{\infty}$ is the endowment bundle for h.

- The Cartesian product of all consumption sets $\mathsf{X} \equiv \prod_{h \in \mathcal{H}} x^h$ gives the aggregate consumption set of the economy.
- Moreover, $\mathbf{x} \equiv \{x^h\}_{h \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \{\omega^h\}_{h \in \mathcal{H}}$ describe the entire consumption allocation and endowments in the economy.
- Each household has a well-defined preference ordering over consumption bundles that can be represented with a real valued utility function: $U^h : X^h \to \mathbb{R}$.
- Assume that U^h is non-decreasing in each of its arguments.
- Let $\mathbf{U} \equiv \{U^h\}_{h \in \mathcal{H}}$ be the set of utility functions.

Production

- Finite number of firms represented by the set of $\mathcal{F}$.
- Each firm $f \in \mathcal{F}$ is characterized by a production set Y^f which specifies the levels that the firm can produce for a given set of inputs.
- An example: only labor and a final good, Y^f would include pairs $(-l, z)$ such that with labor input l the firm can produce at most z.
- We assume that if $y^f \in Y^f$, then $\lambda y^f \in Y^f$ for any $\lambda \in \mathbb{R}_+$

- Let $\mathsf{Y} \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set and $\mathbf{y} \equiv \{y^f\}_{f \in \mathcal{F}}$ such that $y^f \in Y^f$ for all f (or equivalently $\mathbf{y} \in \mathsf{Y}$).
- Finally we need to specify the ownership structure of firms (who gets the potential profits).
- A sequence of profit shares represented by $\boldsymbol{\theta} \equiv \{\theta_f^h\}_{f \in \mathcal{F}, h \in \mathcal{H}}$, such that $\theta_f^h \geq 0$ for all f and h, and $\sum_{h \in \mathcal{H}} \theta_f^h = 1$ for all $f \in \mathcal{F}$.
- Thus, θ_f^h is the share of firm f profits that go to household h.

The definition of an economy

- We can define an economy as $\mathcal{E} \equiv \{\mathcal{H}, \mathcal{F}, \mathsf{U}, \boldsymbol{\omega}, \mathsf{Y}, \mathsf{X}, \boldsymbol{\theta}\}$
- That is for a given set of households and firms an economy is described by preferences, endowments, production sets, consumption sets and an allocation of shares.
- An allocaton for this economy is (x, y) such that x and y are feasible: $\mathsf{x} \in \mathsf{X}$, $\mathsf{y} \in \mathsf{Y}$ and

$$\sum_{h \in \mathcal{H}} x_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} y_j^f,$$

  for all $j \in \mathbb{N}$
- We can no discuss how resources are allocated (or how they should be allocated) in this economy.

Competitive equilibrium

- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$ such that $p_j \geq 0$ for all j.
- One of the goods can be chosen as numeraire (i.e., its price is set to 1).
- As before, $p \cdot z \equiv \sum_{j=0}^{\infty} p_j z_j$.
- A competitive equilibrium for economy $\mathcal{E} \equiv \{\mathcal{H}, \mathcal{F}, \mathsf{U}, \boldsymbol{\omega}, \mathsf{Y}, \mathsf{X}, \boldsymbol{\theta}\}$ is given by an allocation $(\mathsf{x}^* = \{x^{h*}\}_{h \in \mathcal{H}}, \mathsf{y}^* = \{y^{f*}\}_{f \in \mathcal{F}})$ and a price system p^* such that...

1. The allocation is feasible, i.e., $\mathbf{x}^* \in \mathbf{X}$ and $\mathbf{y}^* \in \mathbf{Y}$ and

$$\sum_{h \in \mathcal{H}} x_j^{h*} \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} y_j^{f*},$$

   for all $j \in \mathbb{N}$.
2. For every firm $f \in \mathcal{F}$, y^{f*} maximizes profits:

$$p^* y^f \geq p^* y^f.$$

   for all $y^f \in Y^f$.
3. For every household $h \in \mathcal{H}$, x^{h*} maximizes utility

$$U^h(x^{h*}) \geq U^h(x^h)$$

   for all x such that $x^h \in X^h$ and
   $p^* x^h \leq p^* \cdot \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^f\right).$

- A feasible allocation (x, y) for economy $\mathcal{E} \equiv \{\mathcal{H}, \mathcal{F}, \mathsf{U}, \boldsymbol{\omega}, \mathsf{Y}, \mathsf{X}, \boldsymbol{\theta}\}$ is Pareto optimal if there exists no other feasible allocation $(\hat{\mathsf{x}}, \hat{\mathsf{y}})$ such that $\hat{x^h} \in X^h$ for all $h \in \mathcal{H}$, $\hat{y^f} \in Y^f$ for all $f \in \mathcal{F}$,

$$\sum_{h \in \mathcal{H}} \hat{x}_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} \hat{y}_j^f,$$

for all $j \in \mathbb{N}$ and

$$U^h(\hat{x}^h) \geq U(x^h)$$

for all $h \in \mathcal{H}$ with at least one strict inequality in the previous relationship.

First welfare theorem

- Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv \{\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}\}$
  - with $\mathcal{H}$ finite
  - Assume that all households are locally non-satiated (for all $h \in \mathcal{H}$ at each $x^h \in X^h$, $U^h(x^h)$ is strictly increasing at least one of its arguments and $U^h(x^h) < \infty$)

  Then $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal.
- Proof of the first welfare theorem based on two ideas
  - if another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium
  - any competitive equilibrium already maximizes the set of affordable allocations.

proof by contradiction

- suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $U^h(\hat{x}^h) \geq U^h(x^{h*})$ for all $h \in \mathcal{H}$ and $U^h(\hat{x}^h) > U^h(x^{h*})$ for all $h \in \mathcal{H}'$ where $\mathcal{H}'$ is a non-empty subset of $\mathcal{H}$
- Since $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, it must be that for all $h \in \mathcal{H}$

$$p^* \cdot \hat{x}^h \geq p^* x^{h*} = p^* \cdot \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*}\right) \tag{6}$$

and for all $h \in \mathcal{H}'$

$$p^* \cdot \hat{x}^h > p^* \cdot \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*}\right) \tag{7}$$

- summing (6) over H' and (7) over H' and combining

$$p^* \cdot \sum_{h \in \mathcal{H}} \hat{x}^h > p^* \cdot \sum_{h \in \mathcal{H}} \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*} \right) = p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega^h + \sum_{f \in \mathcal{F}} y^{f*} \right) \tag{8}$$

as sums are finite we can change the order of summation.

- Finally, since $\mathbf{y}^*$ is profit maximizing at prices p^*

$$p^* \cdot \sum_{f \in \mathcal{F}} y^{f*} \geq p^* \cdot \sum_{f \in \mathcal{F}} y^f \tag{9}$$

for any $\{y^f\}_{f \in \mathcal{F}}$ with $y^f \in Y^f$ for all $f \in \mathcal{F}$.

- However, by feasibility of $\hat{x}^h$

$$\sum_{h \in \mathcal{H}} \hat{x}_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} \hat{y}_j^f \tag{10}$$

  for all j.

- Thus,

$$p^* \cdot \sum_{h \in \mathcal{H}} \hat{x}^h \leq p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega^h + \sum_{f \in \mathcal{F}} \hat{y}^f \right) \leq p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega^h + \sum_{f \in \mathcal{F}} \hat{y}^{f*} \right) \tag{11}$$

- This contradicts with (8).

Second welfare theorem

- The converse of first welfare theorem.
- It states that any Pareto optimal allocation can be decentralized as a competitive equilibrium.
- Requires assumptions such as convex consumption and production sets and continuous quasi-concave utility functions +additional technical assumptions (changes in allocations that are very far in the future should not have a large effect).
- The second welfare theorem and a normative representative HH allow us to characterize the optimal growth path that maximize the utility of the representative household and assert that this will correspond to the competitive equilibrium.