Advanced Macroeconomics 1

Oskari Vähämaa (University of Helsinki)

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Lecture 3:
Neoclassical Growth, an introduction

Ramsey model in discrete time (ref: Heer and Maussner, ch 1)

- How much should a country save (Ramsey, 1928)?
- Like the Sollow model but with an endogenous saving rate.
- The most basic DGE model.
- The core of many macroeconomic models.
- Also known as the Ramsey-Cass-Koopmans model or the neoclassical growth model.
- We start with a finite-time version with a constant labor force and no technological growth.
- Let's focus on the planner's problem where the planner maximizes the utility of the representative household

- Time t is divided into intervals of unit length and extends from t=0 to t=T.
- The production function $F(N_t, K_t)$ has the usual properties
    - $F(N, 0) = 0$
    - F is strictly increasing in both of its arguments
    - concave
    - twice continuously differentiable
- The representative agent does not value leisure and seeks to maximize utility function

$$U(C_0, C_1, ..., C_T) \tag{1}$$

- The economy's resource constraint is given by

$$Y_t + (1-\delta)K_t \geq C_t + K_{t+1}, \tag{2}$$

where $0 < \delta < 1$.

- We can simplify the notation

$$f(K_t) \equiv F(N, K_t) + (1-\delta)K_t \tag{3}$$

- The planner's problem is given by

$$\begin{aligned} \max_{\{C_0,...,C_T,K_1,...,K_{T+1}\}} \quad & U(C_0, C_1, ..., C_T) \\ st \quad & \begin{array}{c} K_{t+1} + C_t \leq f(K_t) \\ 0 \leq C_t \\ 0 \leq K_{t+1} \\ for\, all\, t = 0, ..., T \end{array} \end{aligned},$$

where K_0 is given.

- This is a standard non-linear programming problem and thus we can apply the Kuhn-Tucker theorem ($U(\cdot)$ and $f(\cdot)$ are strictly concave, strictly increasing and twice differentiable).
- To proceed, let's write the following Lagrangian:

$$\mathcal{L} = U(C_0, ..., C_T) + \sum_{t=0}^{T} [\lambda_t(f(K_t) - C_t - K_{t+1}) \\ + \mu_t C_t + \omega_{t+1} K_{t+1}]$$

- NB: ω_t are just Lagrangean multipliers here (not the endowments as they were in the previous lecture).

- We get the following FOCs

$$\frac{\partial U(\cdot)}{\partial C_t} - \lambda_t + \mu_t = 0 \qquad t = 0, ..., T$$
$$-\lambda_t + \lambda_{t+1} f'(K_{t+1}) + \omega_{t+1} = 0 \qquad t = 0, ..., T-1$$
$$-\lambda_T + \omega_{T+1} = 0$$
$$\lambda_t(f(K_t) - C_t - K_{t+1}) = 0 \qquad t = 0, ..., T$$
$$\mu_t C_t = 0 \qquad t = 0, ..., T$$
$$\omega_{t+1} K_{t+1} = 0 \qquad t = 0, ..., T,$$

  where $\lambda_t \geq 0, \mu_t \geq 0, \omega_t \geq 0$ for all t.
- As usually, the multipliers value the severeness of the respective constraint (the 'shadow price').
- A constraint that does not bind has a multiplier zero.

- We can use the Inada conditions, $\frac{\partial U(\cdot)}{\partial C_t} \rightarrow \infty$ when $C_t \rightarrow 0 \, \forall t$, to rule out corner solutions $C_t = 0$.
- This implies $\mu_t = 0 \, \forall t$, thus $\frac{\partial U(\cdot)}{\partial C_t} = \lambda_t$
- Moreover, since we assumed $f(0) = 0$, positive consumption also requires $K_t > 0$ from period $t = 0$ through period $t = T$.
- Thus, $\omega_{t+1} = 0$ for $t = \{0, ..., T-1\}$.
- Finally since $U(\cdot)$ is strictly increasing, $f(K_t) - C_t - K_{t+1} = 0 \quad \forall t$

- Putting all pieces together, we get following characterization of optimal solution

$$K_{t+1} + C_t = f(K_t) \tag{4}$$

$$\frac{\frac{\partial U(C_0,\ldots,C_T)}{\partial C_t}}{\frac{\partial U(C_0,\ldots,C_T)}{\partial C_{t+1}}} = f'(K_{t+1}) \tag{5}$$

$$\lambda_T K_{T+1} = 0 \tag{6}$$

- The LHS of (5) is the marginal rate of substitution between two periods and its RHS gives the compensation for an additional unit of saving.

- Since utility is increasing in consumption, $\lambda_T > 0$, and so $K_{T+1} = 0$ (it is optimal to consume everything in the last period).
- The optimal plan is given by

$$K_{t+1} + C_t = f(K_t) \tag{7}$$

$$\frac{\frac{\partial U(C_0,\ldots,C_T)}{\partial C_t}}{\frac{\partial U(C_0,\ldots,C_T)}{\partial C_{t+1}}} = f'(K_{t+1}) \tag{8}$$

and the two boundary conditions, $K_{T+1} = 0$ and initial level of capital K_0

The deterministic infinite-horizon Ramsey model

- Note that, (8) depends on the entire time profile of consumption ($\rightarrow$ need to solve $2T - 1$ simultaneous equations).
- if $T \rightarrow \infty$, we cannot solve this system.
- To circumvent this problem, we restrict to problems that have a recursive structure (i.e., the problems pose themselves in each period the same way)
- For the Ramsey problem, we need to assume that $U(\cdot)$ takes the form stated in lecture 2 eq(1).

- We continue to assume that $\lim_{C\to 0} u'(C) = \infty$, so consumption is always positive and we can ignore $C_t \geq 0$
- The Ramsey problem with infinite time horizon

$$\max_{\{c_0, K_0, \ldots} \sum_{t=0}^{\infty} \beta^t u(C_t)$$
$$C_t + K_{t+1} \leq f(K_t)\,\forall t$$
$$0 \leq K_{t+1}$$
$$K_0 \; given$$

- We can derive the necessary conditions by maximizing the following Lagrangian with respect to $C_0, K_1, C_1, K_2, \ldots$

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(C_t) + \lambda_t (f(K_t) - C_t - K_{t+1}) + \omega_{t+1} K_{t+1}\right]$$

- In order to keep consumption positive, $K_t > 0$ has to hold for all t until "the end of time".
- Moreover, assuming strictly increasing $u(c_t)$ implies $f(K_{t+1}) = C_{t+1} + K_{t+1}$
- Thus, we have the following FOCs

$$u'(C_t) = \lambda_t \tag{9}$$

$$\lambda_t = \beta \lambda_{t+1} f'(K_{t+1}) + \omega_{t+1} \tag{10}$$

$$f(K_t) = C_t + K_{t+1} \tag{11}$$

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} K_{T+1} = 0 \tag{12}$$

- Plugging (9) into (10) and (12)

$$u'(C_t) = \beta f'(K_{t+1}) u'(C_{t+1}) \tag{13}$$

$$C_t + K_{t+1} = f(K_t) \tag{14}$$

$$\lim_{T \to \infty} \beta^T u'(C_T) K_{T+1} = 0 \tag{15}$$

- Eq (13) is referred to the Euler equation for consumption.
- It is equivalent to condition (8) (given the preference specification)
- The flow budget constraint, eq (14) also looks familiar.
- Eq (15) is the so called transversality condition. It is the limit of terminal condition $\lambda_T K_{T+1} = 0$
- It states that the present value of the terminal capital stock must approach zero.

- We can solve C_t from (14) and then plug it into (13) and (15)

$$\frac{u'(f_t(K_t) - K_{t+1})}{u'(f_{t+1}(K_{t+1}) - K_{t+2})} = \beta f'(K_{t+1}) \tag{16}$$

$$\lim_{T \to \infty} \beta^T u'(f(K_T) - K_{T+1}) K_{T+1} = 0 \tag{17}$$

- We have reduced the system of difference equations into a second order difference equation.
- Transversality condition (17) and the initial level of capital, $K(0)$, give us a unique solution to (16).

Transversality condition

- Solve $u'(C_T)$ from (13) (i.e., express $u(C_T)$ with the help of $u(C_{T-1})\, and\, f'(K_T)$) and plug it into (15)

$$0 = \lim_{T\to\infty} \beta^T u'(C_T)K_{T+1} = \lim_{T\to\infty} \beta^T [\frac{u'(C_{T-1})}{\beta f'(K_T)}]K_{T+1}$$

Iterating backwards gives

$$\begin{aligned} 0 &= \lim_{T\to\infty} \beta^T [\frac{u'(C_{T-2})}{\beta^2 f'(K_T) f'(K_{T-1})}]K_{T+1} \\ 0 &= \lim_{T\to\infty} [\frac{u'(C_0)}{\prod_{i=1}^T f'(K_i)}]K_{T+1} \end{aligned}$$

- The transversality condition states that the present value K_{T+1}, measured in the relevant units, has to approach zero when $T \to \infty$.
- If the present value was positive, the consumer would be saving too much. The representative agent could increase her welfare by increasing consumption.

Solving the model

- We have seen that we can characterize the solution with two first order difference equations (or with one second order difference equation) +the transversality condition
- Generally, we can only solve these equations numerically.
- However, sometimes with particular functional forms we can solve the model analytically.
- Moreover, as with the Solow model,we can always solve the steady state analytically.
- It is also possible to analyze transition dynamics graphically (we will postpone this to the week 4)

The steady state

- At a steady state $C_t = C_{t+t}$ and $K_t = K_{t+1}$ for all t.
- Thus,

$$u'(C^*) = \beta f'(K^*) u'(C^*)$$

$$C^* + K^* = f(K^*)$$

or

$$\frac{1}{\beta} = f'(K^*) \tag{18}$$

$$C^* = f(K^*) - K^* \tag{19}$$

An example: a model with an analytical solution

- Assume that the periodic utility function takes a quadratic form
$$u(C_t) = u_1 C_t - \frac{u_2}{2} C_t^2,$$
where $u_1, u_2 > 0$ and that the production function takes a linear form
$$f(K_t) = AK_t,\ A > 0.$$
- Now (13) and (14) can be written as

$$C_{t+1} = \frac{u_1}{u_2}(1 - \frac{1}{\beta A}) + \frac{1}{\beta A} C_t \tag{20}$$

$$K_{t+1} = AK_t - C_t \tag{21}$$

- We will use the method of undetermined coefficients.
- Guess that the consumption, C_t, is a linear function of current capital stock K_t

$$C_t = c_1 + c_2 K_t \tag{22}$$

- If this guess is valid, it has to be consistent with (20), (21) and the transversality condition.
- Substituting (22) into (20) gives

$$c_1 + c_2 K_{t+1} = \frac{u_1}{u_2}\left(1 - \frac{1}{\beta A}\right) + \frac{1}{\beta A}(c_1 + c_2 K_t)$$

$$c_1 + c_2(AK_t - c_1 - c_2 K_t) = \frac{u_1}{u_2}\left(1 - \frac{1}{\beta A}\right) + \frac{1}{\beta A}(c_1 + c_2 K_t)$$

- The last equation holds for arbitrary values of K_t if constant terms on both sides sum to zero

$$0 = c_1 \left(1 - c_2 - \frac{1}{\beta A}\right) - \frac{u_1}{u_2}\left(1 - \frac{1}{\beta A}\right) \tag{23}$$

and if also the coefficients of variable K_t sum to zero

$$0 = c_2 A - c_2^2 - \frac{1}{\beta A} c_2$$

$$c_2 = A - \frac{1}{\beta A} \tag{24}$$

- One can solve c_1 by inserting (24) into (23).

- Next, we can plug our consumption policy into (21)

$$K_{t+1} = \frac{1}{\beta A} K_t - c_1 \tag{25}$$

- If $1/\beta < A$, the capital stock approaches the stationary solution

$$K^* = -\frac{c_1}{1 - \frac{1}{\beta A}} \tag{26}$$

from any given intial value K_0.
- The consumption converges to

$$C^* = \frac{u_1}{u_2} \tag{27}$$

- Together these imply that also transversality condition (15) holds.

Summary of the simple Ramsey model in discrete time

- The planner's problem is

$$\max_{\{c_0,K_0,\ldots} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$C_t + K_{t+1} \leq F(K_t) + (1-\delta)K_t \,\forall t$$

$$0 \leq C_t$$

$$0 \leq K_{t+1}$$

$$K_0 = \hat{K}$$

- Assuming $\frac{\partial U(\cdot)}{\partial C_t} \to \infty$ when $C_t \to 0 \,\forall t$, implies that we can "ignore" the constraint $0 \leq C_t$ and $0 \leq K_{t+1}$ as long as we remember to add the transversality condition (TVC)
- Moreover, since $u(\cdot)$ is strictly increasing in c,

$$C_t + K_{t+1} = F(K_t) + (1-\delta)K_t \,\forall t$$

- Thus, we can just solve the following Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(C_t) + \lambda_t(F(K_t) + (1-\delta)K_t - C_t - K_{t+1})\right] \quad (28)$$

- The first order conditions with respect to C_t and K_{t+1}

$$u'(C_t) = \lambda_t$$

$$\lambda_t = \lambda_{t+1}(F'(K_{t+1}) + (1-\delta))$$

- Moreover, the flow budget constraint

$$C_t + K_{t+1} = F(K_t) + (1-\delta)K_t$$

and TVC: $\lim \beta^T u'(C_T)K_{T+1} = 0$ as $T \to \infty$ have to hold.

- The solution is a sequence $\{C_t, K_{t+1}\}$ such that

$$u'(C_t) = \beta(F'(K_{t+1}) + (1-\delta))u'(C_{t+1}) \tag{29}$$

$$C_t + K_{t+1} = F(K_t) + (1-\delta)K_t \tag{30}$$

for all t, and

$$\lim \beta^T u'(C_T)K_{T+1} = 0, \qquad T \to \infty \tag{31}$$

and $K_0 = \hat{K}$
- NB: we ditched the notation $f(K) \equiv F(K) - (1-\delta)K$.

Numerical solutions (ref: Heer and Maussner, ch3)

Finite-horizon models

- Solving T period problem, we need to find a sequence of capital such that

$$U'(F(K_t + (1-\delta)K_t - K_{t+1}) = \beta(F'(K_{t+1}) + (1-\delta)) * u'(F(K_{t+1} + (1-\delta)K_{t+1} - K_{t+2})$$

holds for $t = 0, ..., T-1$ and $K_{T+1} = 0$

- For a given capital stock K_0, this is a system of T unknown variables, $K_1, ..., K_T$ and T non-linear equations.
- We can use a non-linear equation solver to get a numerical solution.

Non-linear solvers

- Suppose you want to find an x such that $\mathsf{G}(\mathsf{x}) = \mathsf{0}$.
- A typical structure for non-linear solvers is the following iterative scheme:

$$\mathsf{x}^{s+1} = \mathsf{x}^{s} + \mu \Delta \mathsf{x}^{s}, \qquad s = 0, 1, ...$$

- The solvers start with an initial guess of the solution , x^0, determine a direction of change, $\Delta\mathsf{x}$, and a step length μ, then proceed to the next guess of the solution, x^1.
- This process continues until either $\mathsf{G}(\mathsf{x}^{\mathsf{s}}) \approx 0$ or $\mathsf{x}^{s+1} \approx \mathsf{x}^{s}$.

- How to choose the initial guess $\mathbf{x}^0 = (K_1^0, K_2^0, ..., K_T^0)$?
- Need to make sure that consumption stays positive in each period.
- You can try to set K equal to a fraction of $F(K_0) + (1 - \delta)K_0$ for each period.
- The next problem is that the algorithm may end up selecting a point where consumption is negative. If, for example, $u(c) = \ln(c)$, this will result in an error message.
- One solution is to insert a penalty function that hopefully prevents the algorithm from choosing these points.

Infinite-horizon models

- The approach used to solve finite-horizon models can be generalized to infinite-horizon models by taking advantage of the models' property to approach a steady state.
- We just replace $K_{T+1} = 0$ with $K_{T+1} = K^*$.
- Given that there is a stable steady state, the model will converge to that from arbitrary initial conditions.
- practical problems:
    - How to choose T?
    - How to choose the initial guess, i.e., $(K_1^0, ..., K_T^0, K^*)$?

How to choose T

- T has to be large enough so that K_{T+1} is close to K^*
- A method for choosing T:
    1. Start with some small T and solve the model.
    2. Then increase T to T' and solve this larger system.
    3. Compare the first T elements of the latter solution to those of the first solution. if the two solutions are close, T is large enough.
    4. Otherwise, increase T and return to step 1.

Choosing the initial value

- The approach used previously does not necessarily work. If K_0 is small relative to K^*, C_T would be negative.
- Another easy solution would be to set $K_t^0 = K^*$ for $t = 1, ..., T$. However, now it could be that $C_0 < 0$ if K_0 is small enough.
- A method for selecting the initial value:
    1. Choose K_0' that is close to K^* and use a non-linear solver to get a solution to this problem
    2. Use this solution as a starting value for smaller K_0'
    3. Continue this until K_0' has reached K_0.