Advanced Macroeconomics 1

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Lecture 9:
Overlapping generations

- A key feature of the neoclassical growth model is that it admits a representative household.
- However, for many interesting questions this is not an appropriate assumption.
- For example, if we want to understand the interaction between different generations, we need to deviate from the assumption of a representative household.
- Today we will analyze an economy in which new households arrive (are born) over time.
- An important feature is that decisions made by older generations will affect the prices faced by the unborn generations.
- The welfare theorems need not applyin the OLG model.

The Baseline OLG model (ref: Acemoglu ch 9.2-9.3)

Demographics, preferences and technology

- Time is discrete and runs to infinity.
- Each individual lives for two periods. For example, all individuals born at time t live for dates t and t+1
- Let's assume general utility function for individuals born at any date t

$$U_t(c_1(t), c_2(t+1)) = u(c_1(t)) + \beta u(c_2(t+1)), \qquad (1)$$

  where $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the same conditions than before (see, e.g., lecture 7 slide 4).
- In (1) $c_1(t)$ denotes the consumption of an individual born at time t when young (at time t), and $c_2(t+1)$ is this individual's consumption when old (at time $t+1$).
- The discount factor $\beta \in (0, 1)$.

- Factor markets are competitive.
- Individuals
    - can only work in the first period of their lives
    - supply one unit of labor inelastically
    - earn the equilibrium wage $w(t)$
- Population is growing exponentially, i.e.,

$$L(t) = (1+n)^t L(0). \tag{2}$$

- A representative firm with the following production function

$$Y(t) = F(K(t), L(t)) \tag{3}$$

- Let's assume that $\delta = 1$ (full depreciation)
- Again defining $k \equiv K/L$, means that we have

$$y(t) = f(k(t)) = F(K, 1), \tag{4}$$

$$1 + r(t) = R(t) = f'(k(t)), \tag{5}$$

$$w(t) = f(k(t)) - k(t)f'(k(t)). \tag{6}$$

Consumption decisions

- Savings by an individual of generation t, $s(t)$, are determined as a solution to the following problem

$$\max_{c_1(t),c_2(t+1),s(t)} u(c_1(t)) + \beta u(c_2(t+1)),$$

subject to

$$c_1(t) + s(t) \leq w(t) \tag{7}$$

and

$$c_2(t+1) \leq R(t+1)s(t) \tag{8}$$

- Since $u(\cdot)$ is strictly increasing, both constraints hold as equalities, and therefore the FOC for a maximum can be written as

$$u'(c_1(t)) = \beta R(t+1)u'(c_2(t+1)). \tag{9}$$

- Combining (9) with the budget constraints, we obtain the following implicit function that determines savings per person as

$$s(t) = s((w(t), R(t+1)), \tag{10}$$

  where $s : \mathbb{R}^2_+ \to \mathbb{R}$ is strictly increasing in $w(t)$ and and may be increasing or decreasing in its second argument.
- Total savings in this economy are equal to

$$S(t) = s(t)L(t). \tag{11}$$

- Since capital depreciates fully after use, the law of motion of the capital stock is

$$K(t+1) = L(t)s(w(t), R(t+1)). \tag{12}$$

Equilibrium

- **Competitive equilibrium:** A competitive equilibrium can be represented by sequences of aggregate capital stock, households' consumptions, and factor prices, $\{K(t), c_1(t), c_2(t), R(t), w(t)\}_{t=0}^{\infty}$, such that the factor price sequence, $\{R(t), w(t)\}_{t=0}^{\infty}$ is given by (5) and (6), individual consumption decisions, $\{c_1(t), c_2(t)\}_{t=0}^{\infty}$, are given by (7), (8) and (9) and the aggregate capital stock, $\{K(t)\}_{t=0}^{\infty}$, evolves according to (12).
- A steady state equilibrium can be defined in the usual fashion as an equilibrium in which the capital-labor ratio is constant.

- To characterize the equilibrium divide (12) by $L(t+1) = (1+n)L(t)$, to obtain the capital-labor ratio as

$$k(t+1) = \frac{s(w(t), R(t+1))}{1+n} \tag{13}$$

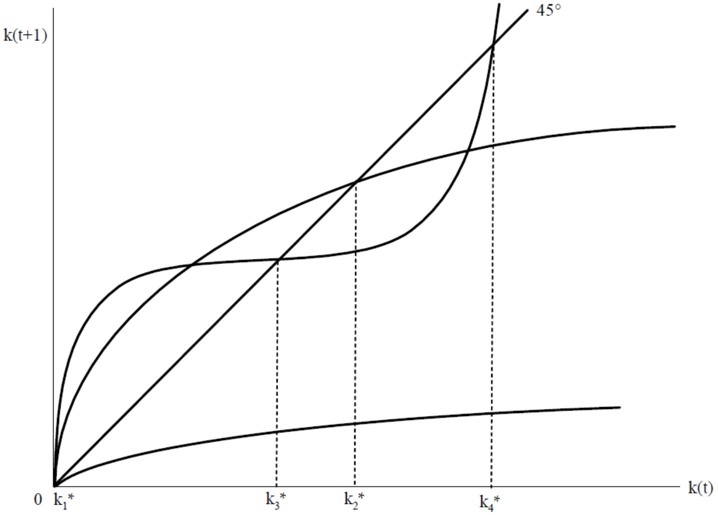
- Substitute for $R(t+1)$ and $w(t)$ from (5) and (6)

$$k(t+1) = \frac{s(f(k(t) - k(t)f'(k(t)), f'(k(t+1))}{1+n} \tag{14}$$

- This is the fundamental law of motion of the OLG economy.
- A steady state is given by

$$k^* = \frac{s(f(k^*) - k^*f'(k^*), f'(k^*)}{1+n} \tag{15}$$

- Since $s(\cdot)$ can take any form, (14) can lead to complicated dynamics. Moreover, there can be a unique steady state or multiple steady states.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Restrictions on utility and production functions

- Suppose that the utility function takes the CRRA form

$$U_t(c_1(t), c_2(t+1)) = \frac{c_1(t)^{1-\theta} - 1}{1-\theta} + \beta \frac{c_2(t+1)^{1-\theta} - 1}{1-\theta}, \quad (16)$$

where $\theta > 0$ and $0 < \beta < 1$.

- Furthermore, assume that technology is Cobb-Douglas

$$f(k) = k^{\alpha}. \quad (17)$$

- With CRRA utility the first order condition is given by

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta}. \quad (18)$$

- The budget constraints imply that

$$c_1(t) = w(t) - s(t) \quad (19)$$

$$c_2(t+1) = R(t+1)s(t). \quad (20)$$

- Plugging (19) and (20) into (18) gives

$$s(t)^{-\theta}\beta R(t+1)^{1-\theta} = (w(t) - s(t))^{-\theta}$$

$$s(t) = \frac{w(t)}{\psi(t+1)}, \qquad (21)$$

where

$$\psi(t+1) \equiv [1 + \beta^{-1/\theta}R(t+1)^{-(1-\theta)/\theta}] > 1,$$

which ensures that savings are always less than earnings.

- The impact of factor prices on savings is summarized by the following derivatives

$$s_w = \frac{1}{\psi(t+1)} \in (0,1)$$

$$s_R = \frac{1-\theta}{\theta}(\beta R(t+1))^{-1/\theta}\frac{s(t)}{\psi(t+1)}$$

- Since $\psi(t+1) > 1$, we have $0 < s_w < 1$.
- Moreover, in this case $S_R < 0$ if $\theta > 1$ and $S_R > 0$ if $\theta < 1$.
- The relationship between the rate of return on savings and the level of savings reflects the counteracting influences of income and substitution effects.

- Substitution effect: when R increases, consumption today becomes more expensive (the price of tomorrow's good is $1/R$) $\rightarrow$ $c_1 \downarrow$ and $c_2 \uparrow$.
- Income effect: agent's capital income increases $\rightarrow c_1 \uparrow$ and $c_2 \uparrow$ (if the agent would be a borrower, the effects would be the opposite).
- When $\theta > 1$, the income effect dominates.
- In contrast, when $\theta < 1$, the substitution effect dominates
- when $u(c) = log(c)$, the two effects are exactly equal.

- Plugging (21) into (14) gives

$$\begin{aligned} k(t+1) &= \frac{s(t)}{1+n} \\ &= \frac{w(t)}{(1+n)\psi(t+1)} \\ &= \frac{f(k(t)) - k(t)f'(k(t))}{(1+n)[1+\beta^{-1/\theta}(f'(k+1))^{-(1-\theta)/\theta}]} \end{aligned}$$

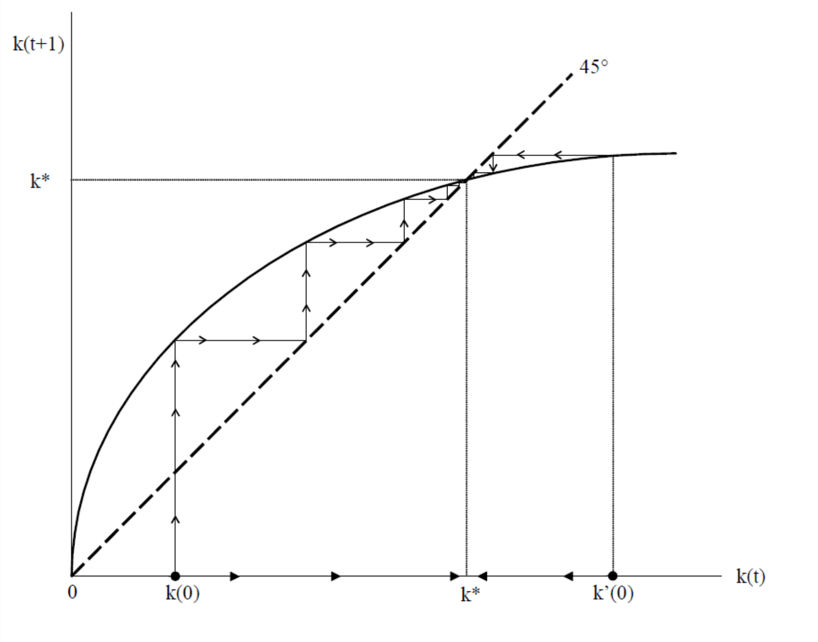
- Using the Cobb-Douglas production function

$$k(t+1) = \frac{(1-\alpha)k(t)^{\alpha}}{(1+n)[1+\beta^{-1/\theta}(\alpha k(t+1)^{\alpha-1})^{-(1-\theta)/\theta}]} \tag{22}$$

- Eq(22) converges to a steady state given by

$$k^* = \frac{f(k^*) - k^* f'(k^*)}{(1+n)[1+\beta^{-1/\theta}(f'(k^*))^{-(1-\theta)/\theta}]} \tag{23}$$

- In this particular case, the equilibrium dynamics are very similar to those of the Solow model.


k(t+1)
45°
k*
0
k(0)
k*
k'(0)
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- Suppose that the utility function is given by

$$U_t(c_1(t), c_2(t+1)) = log(c_1(t)) + \beta log(c_2(t+1)) \qquad (24)$$

- The production technology is Cobb-Douglas, $f(k) = k^\alpha$.
- The Euler equation:

$$\frac{c_2(t+1)}{c_1(t)} = \beta R(t+1), \qquad (25)$$

and it implies that savings should satisfy

$$s(t) = \frac{\beta}{1+\beta} w(t) \qquad (26)$$

- A constant saving rate. (Solow!)

- Combining (26) and with the capital accumulation equation (14)

$$k(t+1) = \frac{\beta w(t)}{(1+n)(1+\beta)} = \frac{\beta(1-\alpha)k(t)^{\alpha}}{(1+n)(1+\beta)}$$

- There exists a unique steady state given by

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}} \tag{27}$$

- Moreover, starting from any $k(0) > 0$, the equilibrium dynamics are identical to the basic Solow model.

Overaccumulation, Pareto optimality and the role of social security (ref: Acemoglu ch 9.4-9.5)

Planner's problem

- Let's return to the general problem and compare the competitive equilibrium to the choice of a social planner wishing to maximize the weighted average of all generations' utilities.
- the planner maximizes

$$\sum_{t=0}^{\infty} \xi_t U_t(c_1(t), c_2(t+1)),$$

where ξ_t is the weight that the planner places on the utility of generation t (with the assumption that $\sum_{t=0}^{\infty} \xi_t < \infty$)

- When $U(\cdot)$ is given by (1), the planner's problem problem is

$$\max \sum_{t=0}^{\infty} \xi_t(u(c_1(t)) + \beta u(c_2(t+1))),$$

subject to the resource constraint

$$F(K(t), L(t)) = K(t+1) + L(t)c_1(t) + L(t-1)c_2(t)$$
$$f(k(t)) = (1+n)k(t+1) + c_1(t) + \frac{c_2(t)}{1+n}$$

- We have the following first order condition

$$u'(c_1(t)) = \beta f'(k(t+1))u'(c_2(t+1))$$

- Since $R(t+1) = f'(k(t+1))$ this equation is identical to (9), i.e., the planner allocates the consumption of a given individual in exactly the same way as the individual himself.

- However, the planner's allocation across generations differs from that in the competitive equilibrium, since the planner is giving different weights to different generations.
- Is the competitive EQM Pareto optimal?
- In general, the answer is no.
- Suppose that the steady state level of capital, k^* is greater than k_{gold} (in the OLG economy there is no reason why this could not be the case).
- In the steady state of the OLG economy

$$f(k^*) - (1+n)k^* = c_1^* + (1+n)^{-1}c_2^* \equiv c^*, \tag{28}$$

where the (first) equation follows by national income accounting.

- Therefore,

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n), \tag{29}$$

  and the golden rule capital-labor ratio is defined by

$$f'(k_{gold}) = 1 + n. \tag{30}$$

- Now if $k^* > k_{gold}$, then $\frac{\partial c^*}{\partial k^*} < 0$, so reducing savings can increase total consumption.
- If this is the case, the economy is said to be **dynamically inefficient**.
- Alternatively, we can define dynamical inefficiency as

$$r^* < n.$$

- Suppose we start from a steady state at time T with $k^* > k_{gold}$
- Consider a variation in which the capital stock is reduced by a small amount Δk, where $\Delta k \in (0, k^* - k_{gold})$, from the next period onwards.
- Then the following changes occur

$$\Delta c(T) = (1+n)\Delta k > 0$$

and

$$\Delta c(t) = -(f'(k^* - \Delta k) - (1+n))\Delta k \qquad for\, all\, t > T.$$

- The first expression gives the direct effect and the second reflects the fact that in addition to the direct effect there is less capital and thus less to consume from $T+1$ onwards.
- For small Δk, $(f'(k^* - \Delta k) - (1+n)) < 0$ and $\Delta c > 0$ for all $t \rightarrow$ Pareto improvement.

Pecuniary externalities

- are the price-related effects of the trading decisions of others on the utility of a household.
- are the reason for the OLG economy potentially being dynamically inefficient.
- Dynamic inefficiency arises from overaccumulation of assets which results from the need of current young generation to save for old age.
- However the more they save, the lower is the interest rate and this may encourage them to save even more.
- An alternative way of providing consumption to individuals in old age might lead to a Pareto improvement.

Social security 1: fully funded

- In fully founded social security system, the government raises amount $d(t)$ from the young.
- These funds are invested in the only productive asset in the economy, the capital stock.
- The workers receive the return when old.
- The individuals maximization problem:

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

$$c_2(t+1) \leq R(t+1)(s(t) + d(t))$$

for a given $d(t)$.

- It is no longer the case that individuals would always choose $s(t) > 0$.
- Two alternative assumptions:
    1. $s(t) \geq 0$ for all t
    2. $s(t)$ is free
- When $s(t)$ is free, the competitive equilibrium applies regardless of a feasible social security plan, $\{d_t\}_{t=0}^{\infty}$.
- The competitive EQM also applies for the case $s(t) \geq 0$ if, for a given sequence $\{d_t\}_{t=0}^{\infty}$, $s(t) > 0$ is optimal for all t.
- Even with $s(t) \geq 0$ fully funded social security cannot lead to a Pareto improvement.

Social security 2: unfunded

- With unfunded social security government collects $d(t)$ from the young at t and distributes it to the current old with per capita transfers $b(t) = (1+n)d(t)$
- The HH's problem

$$\max_{c_1(t),c_2(t+1),s(t)} u(c_1(t)) + \beta u(c_2(t+1)) \tag{31}$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

$$c_2(t+1) \leq R(t+1)s(t) + (1+n)d(t+1).$$

for a given feasible sequence of social security, $\{d_t\}_{t=0}^{\infty}$.

- The rate of return on social security payments is n rather than $r(t+1) = R(t+1) - 1$
- Since only $s(t)$ goes into capital accumulation, unfunded social security discourages savings.
- In the current context reducing savings may lead to a Pareto improvement.
- Suppose individuals of generation t could choose $d(t)$. Whatever they contribute is given to current old and they receive $(1+n)d(t)$ in the next period.
- In this case there would be no savings until $r(t+1) \geq n$.
- Thus, the unfunded social security system would increase interest rate enough that the economy would no longer be dynamically inefficient.

- Note that the government is essentially running a Ponzi game.
- We have a Pareto improving pyramid scheme here.
- When $r^* < n$, the economy allows a range of welfare improving bubbles that can play the same role as unfunded social security.
- We have bubble when an asset trades at a higher value than its intrinsic value.
- Here the maximum rate of return on bubble is n.
- When there is dynamic inefficiency and $r < n$, a bubble provides a better way of transferring resources across time than capital.
- The most famous example is fiat money.