Advanced Macroeconomics 1

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Lecture 5:
More on dynamic programming

Dynamic Programming: an example

(ref: Acemoglu ch6.6)

Saving problem

- Consider the problem of infinitely-lived consumer with the following instantaneous utility function: $u(c)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, continuously differentiable, and strictly concave.
- The consumer
  - discounts the future with the constant discount factor $\beta \in (0, 1)$
  - faces a certain (non-negative) labor income stream of $\{w_t\}_{t=0}^{\infty}$
  - starts life with a given amount of assets $a_0 \in \mathbb{R}$
  - receives a constant net rate of interest $r > 0$ (the gross rate is $1 + r$)
- Let's simplify this general structure and assume that the income stream is constant, i.e., $w_t = w \in \mathbb{R}_+$

- The consumer's problem is

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a_{t+1} = (1+r)a_t + w - c_t$$

with a_o given
- Without further constraints this problem is not well defined. The consumer can build up debt without limit ($\lim_{t\to\infty} a_t = -\infty$).
- This scenario is called a "Ponzi game". It involves the consumer continuously borrowing and rolling over debt.
- The consumer would prefer paths that allow this.

- From an economic point of view these solutions would be absurd (the lender would incur very large losses).
- We need to impose an appropriate budget constraint on the consumer. The flow budget constraint is not sufficient to capture the lifetime budget constraint.
- Solutions:
  - a no-ponzi condition, e.g., $\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T+1}a_{T+1} \geq 0$
  - no borrowing, $a_{t+1} \geq 0$ for all t
  - the natural borrowing limit, $a_{t+1} \geq \underline{a} \equiv -\frac{w}{r}$ for all t.
- Let's assume that the consumer cannot borrow more than he can repay (i.e, $a_{t+1} \geq \underline{a}$).

- However, even with the borrowing limit assets do not have to belong to a compact set.
- A solution:
    - choose some upper bound, $\bar{a}$ and restrict a to lie in the interval $[\underline{a}, \bar{a}]$.
    - Solve the problem and check that a is in the interior of the set.
- In this model, we know that the net present value of consumer's income stream is $\frac{w}{r}$. Thus, set $\bar{a} \equiv \frac{w}{r}$
- Now we can write the consumer's problem in recursive form.
- What state and control variables do we have?

- The only state variable is a_t
- Consumption is given by $c_t = (1+r)a_t + w - a_{t+1}$. Thus, a_{t+1} is the only control variable.
- We have the following Bellman equation

$$V(a) = \max_{a' \in [\underline{a}, \bar{a}]} u((1+r)a + w - a') + \beta V(a') \tag{1}$$

- Given our assumptions, all the results shown earlier hold.
- Among other things, $V(a)$ is differentiable and a continuous solution, $a' = \pi(a)$ exists.
- FOC:

$$u'((1+r)a + w - a') = u'(c) = \beta V'(a') \tag{2}$$

- Based on the envelope condition

$$V'(a') = (1+r)u'(c') \tag{3}$$

- Combining (2) and (3) gives the consumer euler equation in the familiar form

$$u'(c) = \beta(1+r)u'(c') \tag{4}$$

- Since $u(\cdot)$ is assumed to be continuously differentiable and strictly concave, $u'(\cdot)$ always exits and is strictly decreasing.
- Therefore, (4) implies
    - if $r = \beta^{-1} - 1, \qquad c = c'$
    - if $r > \beta^{-1} - 1, \qquad c < c'$
    - if $r < \beta^{-1} - 1, \qquad c > c'$

Discrete state space methods (ref: Heer and Maussner ch4 and Adda and Cooper ch3)

VFI

- A short introduction to value function iteration over a discrete grid.
- VFI is an intuitive way of solving dynamic programming problems.
- A brute force approach that always works, but can be slow.
- Replace the original model (with continuous state space) with a model whose state space contains a finite number of points.
- Thus, the value function is a finite dimensional object.
- E.g., if the state space is one dimensional, we only evaluate the value function over n points, $\mathcal{X} = \{x_1, x_2, ..., x_n\}$.

- Let's work with the basic Ramsey model again. We have the following Bellman equation

$$V(K) = \max_{0 \leq K' \leq f(K)} u(f(K) - K') - \beta V(K'), \tag{5}$$

  where $f(K) \equiv F(K) + (1 - \delta)K$
- The optimal sequence converges monotonically towards the stationary solution K^* determined from the eq $\beta f'(k^*) = 1$
- The economy will stay in the interval $[K_0, K^*]$ (or in the interval $[K^*, K_0]$ if $K_0 > K^*$)
- However, the stationary solution of our modified problem is not equal to K^*.
- Thus, set $K_n > K^*$

- Next we need to choose the number of grid points, n, and the intervals between grid points (the simplest solution is to work with equally spaced points).
- The grid is given by a vector $\mathcal{K} = \{K_1, ..., K_n\}$
- The value function is a vector **v** of n elements. Its ith element gives the life-time utility obtained from a sequence of capital that is optimal given the initial capital stock $K_0 = K_i$.
- The associated policy function can be represented by a vector **h** of indices (we force the planner to stay on the grid)
- i denotes the index of $K_i \in \mathcal{K}$.
- $j \in 1, 2, ..., n$ denotes the index of $K' = K_j \in \mathcal{K}$, i.e, the maximizer of the RHS of the Bellman equation.
- Then $h_i = j$.

- The value vector can be determined by iterating over

$$v_i^{s+1} = \max_{K_j \in \mathcal{D}_i} u(f(K_i) - K_j) + \beta v_j^s \tag{6}$$

$$\mathcal{D}_i = \{K \in \mathcal{K} : K \leq f(K_i)\}$$

  for all i=1,2,...,n

- Successive iterations will converge to the solution $\mathbf{v}^*$ of the discrete valued infinite-horizon Ramsey model according to the contraction mapping theorem.

Value function iteration

1. Choose a grid: $\mathcal{K} = \{K_1, K_2, ..., K_N\}$ $K_i < K_j, i < j = 1, 2, ..., N$
2. Initialize the value function: $\forall i$ set
$$v_i^0 = \frac{u(f(K^*) - K^*)}{1 - \beta}$$
3. Compute a new value function and the associated policy function, $\mathbf{v}^1$ and $\mathbf{h}^1$, respectively. That is, solve (6) for each i.
4. Check for convergence: if $d_\infty(\mathbf{v}^0 - \mathbf{v}^1) < \epsilon$ stop, else replace $\mathbf{v}^0$ with $\mathbf{v}^1$ and $\mathbf{h}^0$ with $\mathbf{h}^1$ and return to step 3.