Advanced Macroeconomics 1

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Lecture 8:
Neoclassical growth, further topics

An example of consumption functions (ref: Acemoglu ch 8.2)

Consumption behaviour and the lifetime budget constraint

- In the previous lecture, we showed that the consumer's optimization leads (among other things) to the following consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \tag{1}$$

- This equation, however, does not by itself give us a unique path for consumption (we need a boundary condition).
- To derive that, for a special case $\varepsilon_u(c(t)) = \theta$, we need to combine (1) with the (modified) lifetime budget constraint (+TVC)

- To proceed, note that the average interest rate from 0 to t can be written as

$$\bar{r}(t)) = \frac{1}{t}\int_0^t r(s)ds \tag{2}$$

- Integrating, (1) we get

$$c(t) = c(0)\exp\left(\int_0^t \frac{r(s)-\rho}{\varepsilon_u(c(t))}ds\right) \tag{3}$$

- If $\varepsilon_u c(t)$ is a constant, θ, (CRRA-utility), (3) simplifies to

$$c(t) = c(0)\exp\left(\frac{\bar{r}(t)-\rho}{\theta}t\right) \tag{4}$$

- In the previous lecture, we wrote the lifetime budget constraint as

$$\int_0^T c(t)L(t)\exp\left(\int_t^T r(s)ds\right)dt + \mathcal{A}(T)$$
$$= \int_0^T w(t)L(t)\,exp\left(\int_t^T r(s)ds\right)dt + \mathcal{A}(0)\,exp\left(\int_t^T r(s)ds\right), \tag{5}$$

- We can express this in a more common way by multiplying it by $\exp\left(-\int_0^T r(s)ds\right)$.

$$\int_0^T c(t)L(t)\exp\left(-\int_0^t r(s)ds\right)dt + \exp\left(-\int_0^T r(s)ds\right)\mathcal{A}(T)$$
$$= \int_0^T w(t)L(t)\,exp\left(-\int_0^t r(s)ds\right)dt + \mathcal{A}(0), \tag{6}$$

- Divide everything by $L(0)$ (remember that $L(t) = L(0)\exp(nt)$) and use $\bar{r}(t)t = \int_0^t r(s)ds$,

$$\int_0^T c(t)\exp(-(\bar{r}(t)-n)t)dt + \exp(-(\bar{r}(T)-n)T)a(T) = \int_0^T w(t)\, exp\,(-(\bar{r}(t)-n)t)\, dt + a(0), \tag{7}$$

- Taking the limit $\to \infty$ and using the transversality condition,

$$\lim_{T\to\infty}[a(T)\exp(-(\bar{r}(T)-n)T)] = 0, \tag{8}$$

gives us the following standard expression for the lifetime budget constraint

$$\int_0^\infty c(t)\exp(-(\bar{r}(t)-n)t)dt = a_0 + \int_0^\infty w(t)\exp(-(\bar{r}(t)-n)t)dt \tag{9}$$

Consumption policy

- Substituting eq (4), $c(t) = c(0)\exp\left(\frac{\bar{r}(t)-\rho}{\theta}\right)$, into the budget constraint (9) gives

$$
\begin{aligned}
c(0) = \left[\int_0^\infty \exp\left(\left(\frac{(1-\theta)\bar{r}(t)}{\theta} - \frac{\rho}{\theta} + n\right) t\right) dt\right]^{-1} \\
\times \left[a(0) + \int_0^\infty w(t)\exp(-(\bar{r}(t)-n)t\right]
\end{aligned}
\tag{10}
$$

- (10) together with the Euler equation (eq(1) with $\varepsilon_u(c(t)) = \theta$) gives the entire path of utility maximizing consumption for the household
- The same method can be applied also to discrete time consumption problems.

Optimal growth, steady state and transitional dynamics (ref: Acemoglu ch 8.3-8.5)

Optimal growth

- The planner's problem in continuous time can be written as

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-(\rho - n)t\right)u(c(t))dt$$

subject to

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t). \tag{11}$$

and k(0)>0.

- Writing the current-value Hamiltonian and modifying the necessary conditions will give us the following conditions

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho) \tag{12}$$

$$\dot{k}(t) = f(k(t)) - (\delta + n)k(t) - c(t) \tag{13}$$

- Moreover, following a similar process as the one that delivered eq(27) in lecture 7 and plugging $\mu(t)$ into the transversality condition gives

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (f'(k(s)) - \delta - n)ds\right)\right] = 0. \qquad (14)$$

- Equations (12), (13) and (14) are necessary and sufficient conditions for the optimal growth path
- These are exactly the same conditions that describe the market equilibrium (see equations (30), (31) and (32))
- This is not a surprise, as we already knew that the welfare theorems hold.

Steady state equilibrium

- Steady state requires that $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$.
- If $\dot{c}(t) = 0$ eq (12) implies

$$f'(k^*) = \rho + \delta \tag{15}$$

  which is the equivalent of the discrete-time optimal growth model.
- Remember that we had to assume $\rho > n$ thus,

$$r^* = f'(k^*) - \delta > n \tag{16}$$

  (Piketty's "r>n"-theory)
- Given k^*, the steady state consumption can be solved from eq (13)

$$c^* = f'(k^*) - (n + \delta)k^* \tag{17}$$

Equilibrium dynamics and the saddle path stability

- In the Solow model, the transitional dynamics were given by one differential equation and we had a globally stable solution since for $\dot{x} = g(x(t)) : 0 = g(x^*)$, $g(x) > 0 \, \forall \, x < x^*$ and $g(x) < 0 \, \forall \, x > x^*$.
- Here the equilibrium is determined by a system of two differential equations (12) and (13) and by two boundary conditions (14) and $k(0)$.
- About stability
    - an unstable system: no convergence
    - a globally stable system: multiple equilibria
    - The saddle-path stability: a unique equilibrium

- Saddle path stability in linear systems (informally): For a linear n-dimensional system the number of negative eigenvalues, m, tells that there exists an m-dimensional subspace such that starting from any $\mathbf{x}(0)$ in this subspace, we have a unique solution with $\mathbf{x}(t) \to \mathbf{x}^*$. (See Acemoglu ch 7.8 theorem 7.18)
- The takeaway message: Compare the number of negative eigenvalues, m, to the number of predetermined state variables. If $m =$ number of state variables, we have a unique solution.
- For non-linear systems, we can linearize around the steady state and explore its local stability by applying the same idea(see theorem 7.19 in Acemoglu's book).

- Let's approximate (13) (12) by a first order Taylor expansion around (k^*, c^*)

$$\dot{k} \approx constant + (f'(k^*) - n - \delta)(k - k^*) - (c - c^*)$$

$$\dot{c} \approx constant + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k - k^*)$$

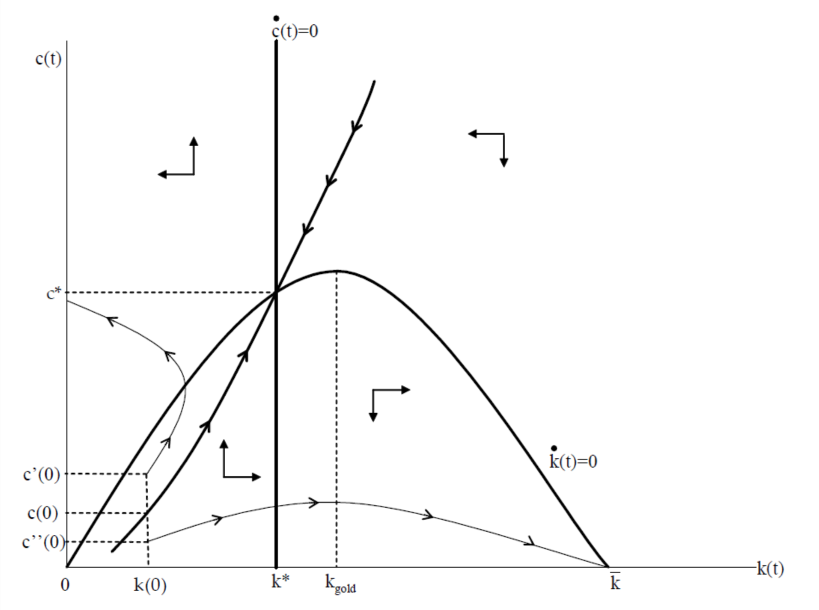
- To find the two eigenvalues (note that $f'(k^*) - \delta = \rho$)

$$\begin{bmatrix} \rho - n - \lambda & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \lambda \end{bmatrix} = 0$$

- Since $\frac{c^* f''(k^*)}{\varepsilon_u(c^*)} < 0$ there are two eigenvalues, one below zero and one above $\Rightarrow$ one dimensional curve, the stable arm, converging to the steady state.
- Note that we have one predetermined variable,k.

Phase diagram

- For 2-dimensional systems there is a more intuitive way to establish uniqueness, by using a graph called phase diagram.
- How to draw a phase diagram:
  - Draw the isoclines, points on $(k(t), c(t))$ space for which $\dot{c}(t) = 0$ or $\dot{k}(t) = 0$.
  - Then, determine how both variables evolve separately in each part of the space (determine the direction of the motion from the difference equation for both variables).
- To ”prove” the uniqueness, rule out all other paths except, for the stable arm.
- One can also use phase diagrams to analyze transitional dynamics.



ċ(t)=0
c(t)
c*
c'(0)
c(0)
c''(0)
k̇(t)=0
0
k(0)
k*
k_{gold}
$\bar{k}$
k(t)

- The stable arm is shown in the figure. For any initial level of capital, consumption jumps to the unique stable path that leads to the steady state (k^*, c^*) (the intersection of the two isoclines)
- After the initial jump, both capital and consumption evolve smoothly according to (12) and (13).
- Uniqueness: All points away from the stable arm will eventually lead to either zero consumption or zero capital.

- If the initial level of consumption was below the stable arm
    - the consumption would reach zero in finite time and capital would accumulate continuously until the maximum level of capital, $\bar{k}$, would be reached (with zero consumption).
    - As $\bar{k} > k_{gold}$ and for $r_{gold} = f'(k_{gold}) - \delta < n$. This path will violate the transversality condition.
- If c(0) were above the stable arm, in this case
    - capital stock would reach zero in finite time, while household consumption would remain positive.
    - This violates feasibility (k cannot be negative and it is never optimal to have a planned jump in consumption)

Technological change and some policy issues (ref: Acemoglu ch 8.7-8.9)

Technological change

- Without exogenous technological change the neoclassical model will converge to a steady state
- Assume that

$$Y(t) = F(K(t), A(t)L(t)), \tag{18}$$

where

$$A(t) = A(0)e^{gt}.$$

- As with the Solow model, we want to redefine variables so that for the new ones we have a well defined steady state.
- Let's define

$$\hat{y}(t) \equiv \frac{Y(t)}{A(t)L(t)} = F(\frac{K(t}{A(t)L(t)}, 1) \equiv f(k(t)), \tag{19}$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)} \tag{20}$$

- In addition to technology, we also need to impose a further assumption on preferences to ensure the balanced growth.
- We need consumption to grow at a constant rate.
- The Euler equation implies that

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho), \tag{21}$$

  if $r(t) \to r^*$, then $\dot{c}(t)/c(t) \to g_c$ is possible only if $\varepsilon_u(\cdot) \to \varepsilon_u$
- That is, the elasticity of marginal utility of consumption has to be asymptotically constant (CRRA-preferences).

- HH's problem is to maximize

$$\int_0^\infty \exp\left(-(\rho - n)t\right)\frac{c(t)^{1-\theta} - 1}{1-\theta}dt, \tag{22}$$

where $c(t) = C(t)/L(t)$, subject to the flow budget constraint and no-Ponzi condition (eq (14) and (15) in lecture 7)

- As before, the Euler equation takes the familiar form

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{23}$$

- However, due to technological progress, output per capita grows and so also $c(t)$ grows.
- Let's define

$$\tilde{c}(t) \equiv \frac{C(t)}{A(t)L(t)} \equiv \frac{c(t)}{A(t)} \tag{24}$$

- This normalized consumption will stay constant along the BGP

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{c}(t)}{c(t)} - g$$
$$= \frac{1}{\theta}(r(t) - \rho - \theta g)$$

- Moreover, the accumulation of capital is given by

$$\dot{k}(t) = f(k(t)) - \tilde{c}(t) - (n + g + \delta)k(t), \quad (25)$$

where recall that $k(t) \equiv K(t)/(A(t)L(t))$

- Finally, the transversality condition can be expressed as

$$\lim_{t\to\infty} \left[k(t) \exp\left(-\int_0^t [f'(k(s)) - g - \delta - n] ds \right) \right] = 0. \quad (26)$$

- The equilibrium interest rate is still given by $r(t) = f'(k(t)) - \delta$.
- Since in steady state (BGP) $\tilde{c}(t)$ must remain constant, $r = \rho + \theta g$, which implies that

$$f'(k^*) = \rho + \delta + \theta g. \tag{27}$$

- This equation determines the steady-state value of the effective capita-labor ratio k^*.
- The level of normalized consumption is given eq (25)

$$\tilde{c}^* = f(k^*) - (n + g + \delta)k^*, \tag{28}$$

while per capita consumption grows at rate g.
- However, there is one additional complication. Now the transversality condition is more demanding

$$\lim_{t\to\infty} \left[k(t) \exp\left(- \int_0^t [\rho - (1-\theta)g - n] ds \right) \right] = 0. \tag{29}$$

- That is, we need to assume that $\rho - n > (1-\theta)g$ for the transversality condition to hold.
- This ensures that households do not achieve infinite utility.
- The steady state effective capital-labor ration is determined endogenously.
- The steady state growth rate is given exogenously and is equal to the rate of labor-augmenting technological progress, g.

The role of policy

- Let's extend the previous framework by introducing linear tax policy.
- Suppose that returns on capital net of depreciation are taxed at rate τ and proceeds are redistributed lumpsum back to households.
- In this case, the capital accumulation is still given by (25), but the net interest rate faced by HH changes to

$$r(t) = (1-\tau)(f'(k(t)) - \delta) \tag{30}$$

- The growth rate of normalized consumption is then obtained from the Euler equation (23) as

$$\begin{aligned} \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} &= \frac{1}{\theta}(r(t) - \rho - \theta g) \\ &= \frac{1}{\theta}((1-\tau)(f'(k(t)) - \delta) - \rho - \theta g) \end{aligned} \tag{31}$$

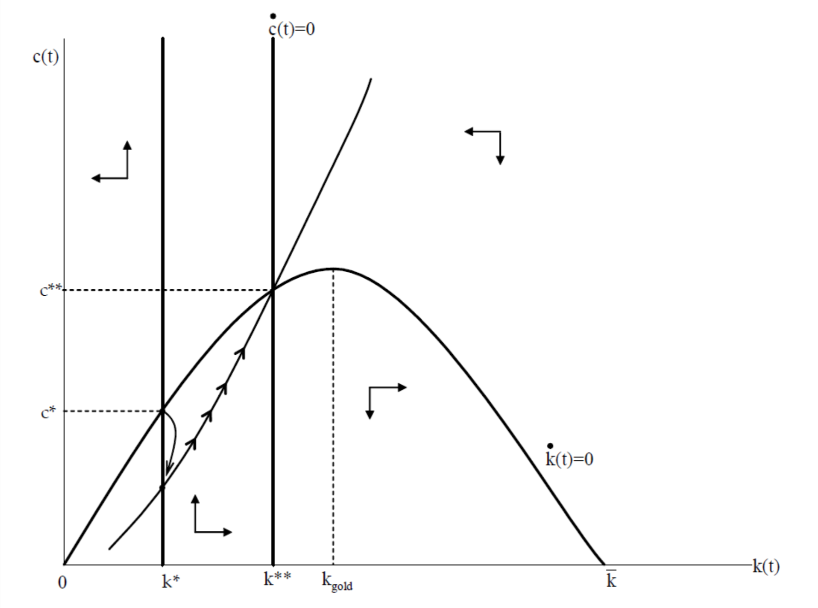
- The steady state capital to effective labor ratio is given by

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau} \tag{32}$$

- A higher tax rate increases the right hand side, and since $f'(\cdot)$ is decreasing, it reduces k^*
- This is one channel through which the policy (and thus institutional) differences might affect economic outcomes.

Comparative dynamics

- How the entire equilibrium path reacts in response to a change in policy or parameters?
- An example: the effects of a change in the tax rate of capital.
- Suppose that population grows at rate n and labor-augmenting technological grows at rate g and that capital is taxed at rate τ.
- Moreover, assume that the economy is initially in steady state $(k^*, \tilde{c}^*)$
- Assume that the capital tax rate declines from τ to τ'.
- How does the equilibrium path change?



ċ(t)=0
c(t)
c**
c*
k̇(t)=0
0
k*
k**
k_gold
k̄
k(t)

A quantitative evaluation

- Consider a world consisting of J closed neoclassical economies.
- Suppose that each country admits a representative HH with identical preferences given by

$$\int_0^{\infty} \exp(-\rho t) \frac{C_j(t)^{1-\theta} - 1}{1-\theta} dt. \tag{33}$$

- Assume that there is no population growth.
- All countries have access to same production technology

$$Y_j(t) = K_j(t)^{\alpha} (AH_j(t))^{1-\alpha}, \tag{34}$$

with H_j representing the exogenously given stock of human capital.

- The accumulation equation is

$$\dot{K}(t) = I_j(t) - \delta K_j(t). \tag{35}$$

- The budget constraint for the HH:

$$(1+\tau_j)I_j(t) + C(t) \leq Y(t), \tag{36}$$

where τ_j is the (constant) country specific tax on investment.

- The competitive EQM can be characterized as the solution to maximization of (33) subject to (35) and (36).
- With the same steps as before, the Euler equation of the representative HH is

$$\frac{\dot{C}_J(t)}{C_j(t)} = \frac{1}{\theta}\left(\frac{\alpha}{(1+\tau_j)}\left(\frac{AH_j(t)}{K_j(t)}\right)^{1-\alpha} - \delta - \rho\right). \tag{37}$$

- At the steady state $\dot{C}_j(t)/C_j(t) = 0$ for all j. Thus the steady state capital for country j is

$$K_j(t) = \left(\frac{\alpha}{(1+\tau_j)(\rho+\delta)}\right)^{\frac{1}{1-\alpha}} AH_j(t). \tag{38}$$

- Countries with higher taxes on investment have a lower capital stock in the steady state.
- Substituting (38) into (34) and comparing two countries with different taxes but the same human capital (and denoting the steady state income level of a country with a tax rate equal to τ by $Y(\tau)$), we obtain

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{\alpha}{1-\alpha}} \tag{39}$$

- We can use (39) to evaluate quantitatively the effects of tax like distortions to income across countries.
- Can neoclassical growth model explain large income differences across countries?
- The answer depends on differences in τ (taxes or wedges) across countries and the value of α.
- How to measure τ?
- One approach is to use the fact that in our model the price of investment goods relative to consumption goods is $1 + \tau$.
- In some countries the relative price of investment goods is almost 8 times as high as in others.

- Assuming that $\alpha = 1/3$ we get from (39) that

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^{0.5} \approx 3. \tag{40}$$

- Thus, differences in taxes or tax-like distortions are unlikely to explain the large income differences (in neoclassical model!).

Neoclassical growth model: a summary

- Compared to the Solow model we have managed to endogenize the saving and consumption policies.
- But we still need an exogenous technological progress to generate growth.
- However, the neoclassical growth model paves the way for further analysis of capital accumulation, human capital investments and endogenous technological progress.
- It is perhaps the most influential model in macroeconomics.