Advanced Macroeconomics 1

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Lecture 7:
Neoclassical growth in continuous time

The neoclassical growth model (ref: Acemoglu ch8)

- We have already seen the planner’s problem in discrete time.
- Now we will focus on the competitive equilibrium in continuous time.
- Today: a model without technological growth.

The basic environment

- Suppose that the economy admits a normative representative household with instantaneous utility function
$$u(c(t)) \tag{1}$$
- We assume that $u(\cdot)$ is defined on $\mathbb{R}_+$ or $\mathbb{R}_+ \setminus \{0\}$ and it is
    - strictly increasing
    - concave
    - twice differentiable

  for all c in the interior of its domain ($u'(c) > 0$, $u''(c) < 0$).
- You may think there is a measure one of households and that each HH has a felicity function given by (1)
- Population within each household grows at rate n, starting with $L(0) = 1$. Thus, the population at time t is
$$L(t) = e^{nt}. \tag{2}$$

- Household is fully altruistic toward all of its future members and always makes the allocations of consumption (among household members) cooperatively
- Then the utility of the household at $t = 0$ can be written as

$$\int_0^\infty e^{-(\rho-n)t}u(c(t))dt, \tag{3}$$

  where $c(t)$ is consumption per capita at t, ρ is the subjective discount rate and the effective discount rate is $\rho - n$. $(e^{-\rho t}e^{nt}u(c_t))$.
- It is assumed that $\rho > n$

- Factor and product markets are competitive and the production function is given by

$$Y(t) = F(K(t), L(t)) \tag{4}$$

- Note that there is no technology term (for now).
- We make the same assumptions than in the lecture 1 (continuity, differentiability, positive and diminishing marginal products, constant returns to scale and that the Inada conditions hold).
- Output per capita is given by

$$y(t) \equiv \frac{Y(t)}{K(t)} = F(\frac{K(t)}{L(t)}, 1) = f(k(t)) \tag{5}$$

  where $k(t) = \frac{K(t)}{L(t)}$.
- As before, competitive factor markets imply

$$R(t) = F_K(K(t), L(t)) = f'(k(t)) \tag{6}$$

$$w(t) = F_L(K(t), L(t)) = f(k(t)) - k(t)f'(k(t)) \tag{7}$$

- Each household decides how to use their assets and allocate consumption over time.
- The flow budget constraint is

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - c(t)L(t), \tag{8}$$

where $\mathcal{A}(t)$ are the asset holdings, $c(t)$ is consumption per capita of the household, $r(t)$ is the risk-free market rate of return and $w(t)L(t)$ is the flow of labor income of the household.
- Define $a(t) \equiv \frac{\mathcal{A}(t)}{L(t)}$. Then, $\frac{\dot{a}(t)}{a(t)} = \frac{\dot{\mathcal{A}}(t)}{\mathcal{A}(t)} - \frac{\dot{L}(t)}{L(t)}$ and

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{9}$$

- At the equilibrium $a(t) = k(t)$ (zero net supply of bonds)
- The market rate of return on assets is then

$$r(t) = R(t) - \delta. \tag{10}$$

Lifetime budget constraint

- We already know that the flow budget constraint is not enough to pin down the lifetime budget constraint.
- The natural debt limit is not useful with sustained growth.
- Let's turn to the no-ponzi condition to overcome this problem.
- The lifetime budget constraint of a HH is

$$\int_0^T c(t)L(t)\exp\left(\int_t^T r(s)ds\right)dt + \mathcal{A}(T)$$
$$= \int_0^T w(t)L(t)\,exp\left(\int_t^T r(s)ds\right)dt + \mathcal{A}(0)\,exp\left(\int_t^T r(s)ds\right), \tag{11}$$

for some arbitrary T.

- This condition states that household's asset position at T is given by its total income plus initial assets minus total consumption, all carried forward to time T.
- Differentiating this wrt T and dividing by L(t) gives (9).
- Assume that (11) hold for finite-horizon economy ending at date T. At this point $\mathcal{A}(T) \geq 0$ has to hold.
- However, the flow constraint (9) does not guarantee this. Need to add $\mathcal{A}(T) \geq 0$ as an additional terminal value constraint.
- In the infinite-horizon case, we need a similar constraint

$$\lim_{t\to\infty} \left[a(t) \exp\left(-\int_0^t (r(s) - n) ds\right) \right] \geq 0. \tag{12}$$

- There is no proper lifetime budget constraint for the representative household without (12).

Definition of equilibrium

- A competitive equilibrium of the Ramsey economy consists of paths of per capita consumption, capital-labor ratio, wage rates and rental rates of capital, $\{c(t), k(t), w(t), R(t)\}_{t=0}^{\infty}$, such that the representative household maximizes (3) subject to (9) and (12) given initial capital-labor ratio $k(0)$ and factor prices $\{w(t), R(t)\}_{t=0}^{\infty}$ with the rate of return on assets $r(t)$ given by (10), and factor prices $\{w(t), R(t)\}_{t=0}^{\infty}$ are given by (6) and (7).

- The household's problem is to maximize

$$\int_0^\infty e^{-(\rho-n)t}u(c(t)), \tag{13}$$

subject to

$$\dot{a}(t) = (r(t)-n)a(t) + w(t) - c(t). \tag{14}$$

$$\lim_{t\to\infty} [a(t)\exp(-\int_0^t (r(s)-n)ds)] \geq 0 \tag{15}$$

and $a(0) = a_0$.

- The current-value Hamiltonian:

$$\hat{H}(t,a,c,\mu) = u(c(t)) + \mu(t)[w(t) + (r(t)-n)a(t) - c(t)], \tag{16}$$

with state variable a, control variable c and current-value costate variable μ.

Necessary conditions

- The necessary conditions for an interior solution are

$$\hat{H}_c(t, a, c, \mu) = u'(c(t)) - \mu(t) = 0 \tag{17}$$

$$\hat{H}_a(t, a, c, \mu) = \mu(t)(r(t) - n) = -\dot{\mu}(t) + (\rho - n)\mu(t), \tag{18}$$

the transition equation

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \tag{19}$$

and the transversality condition

$$\lim_{t\to\infty} \left[\exp\left(-(\rho - n)t\right)\mu(t)a(t)\right] = 0 \tag{20}$$

Sufficiency

- Since $\hat{H}(\cdot)$ is the sum of a concave function of c and a linear function of $(a(t), c(t))$, it is concave in (a, c).
- Thus, to verify that (17)-(20) characterize the optimum, it only remains to show that for any feasible $(a(t), c(t))$, $\lim_{t\to\infty} [\exp(-(\rho - n)t)\mu(t)a(t)] \geq 0$ (see slide 21 in lecture 6)
- This can be rewritten (see eq (27)) as

$$\lim_{t\to\infty} [a(t)\exp(-\int_0^t (r(s) - n)ds)] \geq 0. \tag{21}$$

This is identical the to no-Ponzi condition (12)!

More on necessary conditions

- Rearrange condition (17)

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho). \tag{22}$$

- The multiplier changes depending on whether the rate of return on assets is currently greater than or less than the discount rate of the household.
- Condition (17) implies that

$$u'(c(t)) = \mu(t). \tag{23}$$

- Differentiate (23) wrt t and divide by $\mu(t)$

$$\frac{u''(c(t))c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)} \tag{24}$$

- Substituting this expression into (22) yields the continuous-time consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho), \tag{25}$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))c(t)}{u'(c(t))} \tag{26}$$

- Consumption grows over time when the rate of return on assets is higher than the discount rate.

- $\frac{1}{\varepsilon_u(c(t))}$ is the intertemporal elasticity of substitution. It specifies the speed at which consumption reacts to a gap between $r(t)$ and ρ.
- $\varepsilon_u(c(t))$ measures the curvature (concavity) of the utility function.
- Concavity implies that agents prefer smooth consumption profiles.

Transversality condition

- Integrate eq(22)

$$\begin{aligned}\mu(t) &= \mu(0)\exp\left(-\int_0^t (r(s)-\rho)ds\right) \\ &= u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)ds\right)\end{aligned} \tag{27}$$

- Substitute (27) into the transversality condition (20)

$$\begin{aligned}\lim_{t\to\infty}\left[exp(-(\rho-n)t)a(t)u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)ds\right)\right] = 0 \\ \lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s)-n)ds\right)\right] = 0\end{aligned} \tag{28}$$

- The No-ponzi condition holds as equality.
- $\rightarrow$ The lifetime budget constraint holds as equality.
- This is an implication of local non-satiation.
- Transversality condition ensures that the consumer uses resources efficiently even very far in the future.
- Since at the equilibrium $a(t) = k(t)$, (28) can also be written as

$$\lim_{t\to\infty} \left[k(t) \exp\left(-\int_0^t (r(s) - n) ds\right) \right] = 0. \tag{29}$$

- The discounted market value of capital stock in the very far future should be zero.

Equilbrium

- (25), (28) and (19) pin down the unique solution, $(\hat{a}(t), \hat{c}(t))$, to HH's problem.
- At the equilibrium $a(t) = k(t)$
- Moreover, $r(t) = R(t) - \delta = f'(k(t)) - \delta$ and $w(t) = f(k(t)) - f'(k(t))k(t)$.
- Plug these equilibrium conditions into (19)

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

$$\dot{k}(t) = f(k(t)) - (\delta + n)k(t) - c(t) \tag{30}$$

- We can also rewrite the Euler equation (25) as

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho) \tag{31}$$

- Finally also the transversality condition can expresses as

$$\lim_{t\to\infty} \left[k(t) \exp\left(-\int_0^t (f'(k(s)) - \delta - n) ds\right)\right] = 0. \tag{32}$$

- The equilibrium can be summarized by two differential equations (30) and (31) and by two boundary conditions (32) and k_0.