Advanced Macroeconomics 1

Oskari Vähämaa (University of Helsinki)

September 17, 2019

Lecture 4:
Dynamic programming

Infinite-horizon optimization and dynamic programming (ref: Acemoglu, ch 1 and Adda and Cooper, ch2)

- Let's start with the finite-time Ramsey problem that we analyzed in the previous lecture

$$\max_{\{C_0, K_1, \ldots, K_{T+1}\}} \sum_{t=0}^{T} \beta^t u(C_t)$$

$$C_t + K_{t+1} = f(K_t)$$

$$K_{t+1} \geq 0$$

, where $f(K) \equiv F(K) + (1 - \delta)K$

- Instead of writing the Lagrangian, we can solve this recursively.
- As we know, at period T it is optimal to consume everything.
- Let's note the value associated with this policy as $V_T(K_t) = u(f(K_T))$.

- Using this, we can write the planner's problem at period $T-1$ as

$$V_{T-1}(K_{T-1}) = \max_{K_T} u(f(K_{T-1}) - K_T) + \beta V_T(K_T).$$

- We can continue this up to t=0.
- Thus, for period t, the planner's problem would look like

$$V_t(K_t) = \max_{K_{t+1}} u(f(K_t) - K_{t+1}) + \beta V_{t+1}(K_{t+1}). \tag{1}$$

- For each period, we can think of the problem as a two-period one where everything that relates to the future is embedded in V_{t+1}
- The variable K_t is called a state variable, it tells us everything we need to know to make an optimal decision at time t.

Infinite-horizon

- As we see later, it is also possible to write the infinite-horizon problem in the form

$$V(K) = \max_{K'} u(f(K) - K') + \beta V(K'), \tag{2}$$

where ' denotes the next period variables.

- The basic idea of dynamic programming is to turn a problem of finding an infinite sequence into a functional equation.
- That is, our goal is to try to find function V.
- Let's start with a bit of general theory about these problems.

- A typical sequence problem in the state-control form:

$$V(x) = \max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta u(x_t, y_t)$$

st

$$y_t \in \tilde{G}(x_t)$$

$$x_{t+1} = \tilde{f}(x_t, y_t),\ x_0 \ given$$

- Where

$$\beta \in (0,1)$$

$$x_t \in X \subset \mathbb{R}^{K_x} \qquad \text{state variables}$$

$$y_t \in Y \subset \mathbb{R}^{K_y} \qquad \text{control variables}$$

$$u : X \times Y \to \mathbb{R} \qquad \text{instantaneous payoff function}$$

$$\tilde{G} : X \rightrightarrows Y \quad \text{gives the values of control allowed given the state}$$

$$f : X \times Y \to X \qquad \text{transition equation}$$

The sequence problem: state only formulation

- It is often convenient to substitute y_t as a function of x_t and x_{t+1}. The state-only formulation:

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta u(x_t, x_{t+1})$$

$$st\, x_{t+1} \in G(x_t),\ x_0\ given,$$

  where $V : X \rightarrow, \mathbb{R}\ G : X \rightrightarrows Y$ and $u : X \times X \rightarrow \mathbb{R}$.
- Note that the problem is stationary (u and G do not depend on time).
- Now, the control vector is x_{t+1}.

- Assume that a sequence $\{x_t^*\}_{t=0}^{\infty}$ is the solution to the sequential problem stated in the previous slide
- Define the set of feasible sequences (or plans), starting with an initial value x_t as

$$\Phi(x_t) = \{\{x_s\}_{s=t}^{\infty} : x_{s+1} \in G(x_s) \qquad for\, s = t, t+1, ...\}$$

,i.e., $\Phi(x_t)$ is the set of feasible choices of vectors starting from x_t.
- Denote an element of $\Phi(x_0)$ by $\mathbf{x} = (x_0, x_1, ...) \in \Phi(x_0)$.
- **A1**: Assume $G(x)$ is non-empty for all $x \in X$; and for all $x_0 \in X$ and $\mathbf{x} \in \Phi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1})$ exists and is finite.

Principle of optimality

- Given the assumptions in the previous slide and $\mathbf{x}^* \in \Phi(x(0))$, then

$$\begin{aligned} V(x_0) &= \sum_{t=0}^{\infty} \beta^t u(x_t^*, x_{t+1}^*) = u(x_0, x_1^*) + \beta \sum_{t=0}^{\infty} \beta^t u(x_{t+1}^*, x_{t+2}^*) \\ &= u(x_0, x_1^*) + \beta V(x_1^*) \end{aligned}$$

- Whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions.
- As everything is time-independent, this generalises to

$$V(x_t^*) = u(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*) \tag{3}$$

for $t = 0, 1, ...,$ with $x^*(0) = x(0)$

- Moreover, if any $\mathbf{x}^* \in \Phi(x_0)$ satisfies (3), then it attains the optimal value in the sequence problem.

Bellman equation

- We can convert the sequence problem into a recursive formulation of solving (Bellman equation):

$$V(x) = \max_{x' \in G(x)} u(x, x') + \beta V(x') \qquad for\, all\, x \in X, \qquad (4)$$

where x' denotes the next period state.

- Instead of an infinite-time problem, we have a two period problem.
- (4) is recursive as (unknown) V is on both sides (functional equation).

- Solution to (4) is a time-invariant policy function (correspondence?) from current state x into the future state x'.

$$V(x) = u(x, \pi(x)) + \beta V(\pi(x)) \qquad for\,all\,x \in X. \tag{5}$$

- The benefits of recursive formulation
    - better intuition
    - analytical solutions in some special cases
    - easier to solve numerically
    - powerful tools to establish its properties

Additional stationary dynamic programming theorems

- Let's make the following assumptions
- **A2**:
  - X is a compact subset of $\mathbb{R}^K$
  - G is non-empty valued, compact-valued and continuous
  - $u : \mathsf{X}_G \to \mathbb{R}$ is continuous, where $\mathsf{X}_G = \{(x, y) \in X \times X : y \in G(x)\}$.
- **A3**: u is strictly concave and constraint set G is convex.
- **A4**: For each $y \in X$, $u(\cdot, y)$ is strictly increasing in state and G is monotone in the sense that $x \leq x'$ implies $g(x) \subset g(x')$.
- **A5**: u is continuously differentiable on the interior of its domain X_G

- When **A1** and **A2** hold, then there exists a unique continuous and bounded function $V : X \to \mathbb{R}$ that satisfy (4). Moreover, for any $x_0 \in X$ an optimal plan $\mathbf{x}^* \in \Phi(x_0)$ exists.
- If **A1**, **A2** and **A3** hold, then the unique V is strictly concave and there exists a unique optimal plan $\mathbf{x}^* \in \Phi(x_0)$ for all $x_0 \in X$. This can be expressed as $x^*_{t+1} = \pi(x^*_t)$, where $\pi : X \to X$ is a continuous policy function.
- If **A1**, **A2** and **A4**, then V is strictly increasing in all of its arguments.
- If **A1**, **A2**, **A3** and **A5** hold and we assume that $x \in Int\, X$ and $\pi(x) \in Int\, G(x)$, then $V(\cdot)$ is differentiable at x, with gradient given by $DV(x) = D_x u(x, \pi(x))$.

T operator

- A map is like a function, but over functions rather than numbers.
- The RHS of the Bellman equation can be written as a map in functions

$$T(W)(x) = \max_{x' \in G(x)} u(x, x') + \beta W(x') \, \forall x \in X$$

  T takes a (”value”) function and maps it into another (”value”) function.
- Any $V(x)$ such that $V(x) = TV(x)$ for all x solves the Bellman equation.
- Moreover, it is a fixed-point, i.e., $Tf(x) = f(x)$
- We will be talking about on-to maps, that take a certain function space into the same space

- Next, we go through an intuitive sketch for existence and uniqueness
- If we can prove that T is a contraction, we can use the contraction mapping theorem.
- It implies that
    1. there is an unique fixed point such that $V(x)=TV(x)$
    2. this fixed point can be reached by an iteration process starting from an arbitrary initial condition.

Detour: some math preliminaries

- We need to define the size of each function, i.e. we need a norm. Norm $\| \cdot \|$: 1) $\| f \| \geq 0$ and $\| f \| = 0$ iff $f = 0$ 2) $\| \alpha f \| = | \alpha | \| f \|$ 3) $\| f^1 + f^2 \| \leq \| f^1 \| + \| f^2 \|$
- In our case, we can use sup-norm $\| f(x) \|_\infty = sup_{x \in X} | f(x) |$ which implies the following metric

$$d_\infty(f, g) = \sup_{x \in X} | f(x) - g(x) |$$

- The space $(C(X_G)_b, d_\infty)$ is a complete metric space.
- Recall: Euclidian space is complete iff any Cauchy sequence converges to an element of the space. Same here: for any $\varepsilon > 0$ $\exists N_\varepsilon$: $d_\infty(v_n(x) - v_m(x)) \leq \varepsilon$ for $n, m \geq N_\varepsilon \Rightarrow$ there is $\nu \in C(X_G)_b$ such that $v_n(x) \rightarrow \nu(x)$.

Contraction maps

- Contraction map: An on-to map T is a contraction map iff there exists a number $\beta \in [0, 1)$ such that

$$\| Tf_1 - Tf_2 \| \leq \beta \| f_1 - f_2 \|$$

- In English: T contracts the space between two functions, i.e., functions Tf_1 and Tf_2 are closer to each other than f_1 and f_2.

Contraction maps (2)

- Why is this useful?
- Think about a sequence of functions $f_n = Tf_{n-1}$
- If T is a contraction map, then

$$\begin{aligned} \| f_n - f_{n-1} \| &= \| Tf_{n-1} - Tf_{n-2} \| \\ &\leq \beta \| f_{n-1} - f_{n-2} \| \\ &< \| f_{n-1} - f_{n-2} \| \end{aligned}$$

  Functions in the sequence become closer and closer.
- If the function space is complete, the sequence converges to

$$f_n \to f^*$$

  where f^* is a member of the functions space.

Theorem

If $T : M \to M$ is a contraction map and (M,d) is a complete metric space, then T has a unique fixed point. Moreover, for any initial guess f_0, the sequence $f_n = Tf_{n-1}$ converges to that fixed point.

- If we can write a Bellman equation as a fixed point problem of a map and prove that the map is a contraction...
- We know there is a unique solution and we can find it starting from any initial guess by iterating.

Blackwell's sufficient conditions

- In general, it is hard to prove that a map is a contraction. We can often use Blackwell's sufficient conditions for T to be a contraction mapping
    1. Monotonicity: $f_1(x) \leq f_2(x)$ for all x implies that $Tf_1(x) \leq Tf_2(x)$ for all x
    2. Discounting: there exists a $\beta \in [0, 1)$ such that for any constant function c and for any function f, $T(f + c) \leq Tf + \beta c$.

- Now we can proof that the RHS of the Bellman equation

$$T(W)(x) = \max_{x' \in G(x)} u(x, x') + \beta W(x') \, \forall x \in X$$

  is a contraction.

- Monotonicity: assume that $W(x) > Q(x)$ for all x and let $p_Q(x)$ be the policy function obtained from

$$\max_{x' \in G(x)} u(x, x') + \beta Q(x') \, \forall x \in X$$

  Then
  $T(W)(x) = \max_{x' \in G(x)} u(x, x') + \beta W(x') \geq u(x, p_Q(x)) + \beta W(p_q(x)) \geq u(x, p_Q(x)) + \beta Q(p_Q(x)) \equiv T(Q)(s) \, \forall x \in X$

- Discounting:
  $T(W+k) = \max_{x' \in G(x)} u(x,x') + \beta(W(x')+k) = T(W) + \beta k \, \forall x \in X.$
- As $0 < \beta < 1$, T operator is a contraction map.
- and based on the contraction mapping theorem there exists a unique solution to (4).

Numerical DP and contractions

- Make an initial guess for value function $V_0(x)$ for all x, e.g., $v_0(x) = 0$
- Update the guess using T. That is $V_1(x) = TV_0(x)$
- Compare the distance between V_1 and V_0. If $d_\infty(V_0, V_1) \approx 0$, stop.
- Otherwise update, $V_2 = TV_1$ and compare V_2 and V_1.
- Continue iterating until $d_\infty(TV_t, V_t) \approx 0$.

The first order conditions

- Consider the functional equation

$$V(x_t) = \max_{x_{t+1} \in G(x_t)} u(x_t, x_{t+1}) + \beta V(x_{t+1}), \tag{6}$$

- Let's assume that assumptions **A1** $-$ **A5** hold.
- Thus, (6) is strictly concave and the maximand is differentiable.
- The optimal solutions can be characterized by the following Euler equation

$$D_{x_{t+1}} u(x_t, {x_{t+1}}^*) + \beta DV(x^*_{t+1}) = 0, \tag{7}$$

  where $D_{x_{t+1}}$ is a vector of partial derivatives wrt the control vector.
- Or denoting control vector with y

$$D_y u(x_t, y^*) + \beta DV(y^*) = 0 \tag{8}$$

"Envelope theorem"

- But we do not know $V(\cdot)$?
- Notation: $x^*_{t+1} = y^* = \pi(x_t)$ and D_y is the partial derivative vector wrt controls while D_x is the partial derivative vector wrt states.
-

$$
\begin{aligned}
V(x_t) &= \max_{x_{t+1} \in G(x_t)} u(x_t, x_{t+1}) + \beta V(x_{t+1}), \\
V(x_t) &= u(x_t, \pi(x_t)) + \beta V(\pi(x_t)) \\
DV(x) &= D_x u(x_t, \pi(x_t)) + D_y u(x_t, \pi(x_t)) \\
&\quad * D_x \pi(x_t) + \beta DV(\pi(x_t)) * D_x \pi(x_t) \\
&= D_x u(x_t, \pi(x_t)) + \underbrace{[D_y u(x_t, \pi(x_t)) + \beta DV(\pi(x_t)] D_x \pi(x_t)}_{=0 \text{ see eq(8)}}
\end{aligned}
$$

- This holds also for the next period, thus

$$\beta DV(x^*_{t+1}) = \beta D_x u(\pi(x), \pi(\pi(x))) \qquad (9)$$

- Pluging (9) into (8) gives

$$D_y u(x, \pi(x)) + \beta D_x u(\pi(x), \pi(\pi(x))) = 0 \tag{10}$$

or

$$D_y u(x_t^*, x_{t+1}^*) + \beta D_x u(x_{t+1}^*, x_{t+1}^*) = 0 \tag{11}$$

- (11) is the familiar euler equation. Eq (10) states that this can also be written as a functional equation of an unknown policy function.
- To fully characterize the optimum we also need the transversality condition

$$\lim_{t \to \infty} \beta^t D_x u(x_t^*, x_{t+1}^*) x_t^* = 0 \tag{12}$$

An example: Ramsey model

- Consider the following problem

$$\max_{k_{t+1}, c_t} \sum_{t=0}^{\infty} u(c_t)$$

  subject to

$$k_{t+1} + c_t = f(k_t)$$

- The Bellman equation (in the state only form)

$$V(k_t) = \max_{k_{t+1}} u(f(k_t) - k_{t+1}) + \beta V(k_{t+1}) \tag{13}$$

- The first order condition

$$-u'(f(k_t) - k^*_{t+1}) + \beta V'(k^*_{t+1}) = 0 \tag{14}$$

- Use the envelope condition,

$$V'(k^*_{t+1}) = u'(f(k^*_{t+1}) - k^*_{t+2})f'(k^*_{t+1}), \tag{15}$$

to write the FOC as

$$u'(f(k_t) - k^*_{t+1}) = \beta u'(f(k^*_{t+1}) - k^*_{t+2})f'(k^*_{t+1}) \tag{16}$$

Log-preferences and Cobb-Douglas technology

- Let's continue with the previous example and let's assume that $u(c) = \ln c$ and $f(k) = k^\alpha$.
- Equation (16) can be written as

$$\frac{1}{k^\alpha - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^\alpha - \pi(\pi(k))}. \tag{17}$$

  This has to hold for all k.
- Guess that this functional equation can be solved with a policy function that takes the following form

$$\pi(k) = ak^\alpha. \tag{18}$$

  and plug it into (17)

- Thus,

$$\frac{1}{k^\alpha - ak^\alpha} = \beta \frac{\alpha a^{\alpha-1} k^{\alpha(\alpha-1)}}{a^\alpha k^{\alpha^2} - a^{1+\alpha} k^{\alpha^2}}$$
$$= \frac{\beta}{a} \frac{\alpha}{k^\alpha - ak^\alpha}$$

- $a = \beta\alpha$ satisfies this equation.
- No need to worry about the transversality condition as k converges to a steady state.