Example: value function iteration

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- Neoclassical growth model with full capital depreciation, when the felicity function is given by $u(c_t) = \ln c_t$ and the production function, in per capita terms, takes the form: $f(k_t) = k_t^\alpha$ where $0 < \alpha < 1$. Moreover, the aggregate resource constraint $f(k_t) = c_t + k_{t+1}$ has to hold for all periods

- The planner's problem can be written recursively as

$$V(k) = \max_{k'} \ln(k^\alpha - k') + \beta V(k'), \tag{1}$$

  where k' denotes the next period capital (chosen today).

- How to solve this functional equation?

- As

$$T(W)(k) = \max_{k'} \ln(k^\alpha - k') + \beta W(k') \tag{2}$$

  contacts towards the fixed point V for any W in the set of bound and continuous functions, we can construct a sequence by always applying T operator to get the next member in the sequence. The limit of this sequence is the (value) function that solves the (1)

- Let's guess that the value function $V^0 = 0$ for all k.

- Use T operator to get

$$V_1(k) = T(V^0)(k) = \max_{k'} \ln(k^\alpha - k') \tag{3}$$

  Thus, it is optimal to set $k'(k) = 0 \, \forall k$ and so V^1 can be written as

$$V_1(k) = \ln(k^\alpha) = \alpha \ln k \tag{4}$$

- Use T operator again

$$V^2(k) = T(V^1)(k) = \max_{k'} \ln(k^\alpha - k') + \beta\alpha \ln(k') \tag{5}$$

The solution to the two period problem can be found by taking the first order condition wrt k'

$$\frac{1}{k^\alpha - k'(k)} = \frac{\beta\alpha}{k'(k)}$$

$$k'(k) = \frac{\beta\alpha}{1+\beta\alpha} k^\alpha \tag{6}$$

and the optimal consumption (given V^1) is

$$c(k) = \frac{1}{1+\beta\alpha} k^\alpha. \tag{7}$$

Thus, V_2 written without *max* is

$$V_2(k) = \ln(c(k)) + \beta V^1(k'(k)) = \ln\left(\frac{1}{1+\beta\alpha} k^\alpha\right) + \beta\alpha \ln\left(\frac{\beta\alpha}{1+\beta\alpha} k^\alpha\right)$$

$$V^2(k) = \alpha(1+\alpha\beta)\ln k + A_1, \tag{8}$$

where $A_1 \equiv \ln\left(\frac{1}{1+\beta\alpha}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right)$

- The third round:

$$V^3(k) = T(V^2)(k) = \max_{k'} \ln(k^\alpha - k') + \beta\alpha(1+\alpha\beta)\ln(k') + \beta A_1 \tag{9}$$

The first order condition with respect to k' is given by

$$\frac{1}{k^\alpha - k'(k)} = \frac{\beta\alpha(1+\beta\alpha)}{k'(k)}$$

$$k'(k) = \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} k^\alpha \tag{10}$$

and optimal consumption is

$$c(k) = \frac{1}{1+\alpha\beta+(\alpha\beta)^2} k^\alpha \tag{11}$$

Plugging the policies into eq (9) gives

$$V^3(k) = \ln\left(\frac{1}{1+\alpha\beta+(\alpha\beta)^2} k^\alpha\right) + \beta\alpha(1+\alpha\beta)\ln\left(\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} k^\alpha\right) + \beta A_1$$

$$V^3(k) = \alpha(1+\alpha\beta+(\alpha\beta)^2)\ln k + A_2, \tag{12}$$

where $A_2 \equiv \ln\left(\frac{1}{1+\alpha\beta+(\alpha\beta)^2}\right) + (\alpha\beta+(\alpha\beta)^2)\ln\left(\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right)$

- If we continue iterating up to s we have

$$k'(k) = \frac{\sum_{i=1}^{s-1}(\alpha\beta)^i}{\sum_{i=0}^{s-1}(\alpha\beta)^i} k^\alpha. \tag{13}$$

  and so

$$\begin{aligned} k'(k) &= \lim_{s\to\infty} \frac{-1+1+\sum_{i=1}^{s-1}(\alpha\beta)^i}{\sum_{i=0}^{s-1}(\alpha\beta)^i} k^\alpha \\ &= \lim_{s\to\infty} (-\frac{1}{\sum_{i=0}^{s-1}(\alpha\beta)^i} - 1)k^\alpha \end{aligned}$$

$$k'(k) = \lim_{s\to\infty} (\frac{1}{\frac{1}{1-\alpha\beta}} + 1)k^\alpha = \alpha\beta k^\alpha \tag{14}$$

$$c(k) = (1-\alpha\beta)k^\alpha. \tag{15}$$