Heterogeneous price technologies

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Abstract

We present a unified theory of dynamic oligopoly pricing with heterogeneous information technologies on each side of the market. Trackers (on the firm-side) and shoppers (on the consumer-side) can follow market prices costlessly, whereas non-trackers and non-shoppers cannot. We describe both non-collusive and collusive equilibria. While the effects of tracking may be non-monotone, the presence of trackers generally harms consumers. The price pattern that arises with trackers and non-trackers reconciles a multitude of cross-sectional price patterns, such as persistent price differences and adherence to or avoidance of certain prices. We find that non-trackers can counter tracker collusion by applying a limit-price strategy.

Keywords: *Dynamic oligopoly, commitment, price dispersion, price tracking, limit pricing, collusion.* **JEL-codes:** D43, D83, L11, L41.

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1 Introduction

Homogeneous goods and close substitutes are available at persistently dissimilar prices among firms (Baye et al., 2006b,a; Chandra and Tappata, 2011; Kaplan and Menzio, 2015; Gorodnichenko and Talavera, 2017). Classic models of sales, following Varian (1980), provide a partial explanation of price dispersion, based on *consumer-side heterogeneity* of either price information (Bergemann et al., 2021) or price search commitment (Stahl, 1989). Information differences among consumers engender a mixed-price equilibrium in which competing firms randomize between offering high prices, targeting uninformed consumers, and low prices, targeted to informed consumers – while observing neither rival prices nor which consumers they encounter.

Nevertheless, standard approaches struggle to account for persistent price dispersion, as firms are likely to observe which prices their rivals offer over time. Information alters optimal prices by changing firms' prospects of attracting informed consumers. Firms are known to routinely monitor competing prices, through similar channels as consumers use, such as price comparison sites or apps.[1] A survey by the European Commission (EC, 2014) reports that 53 % of retailers track competitor prices. Additionally, evidence suggests firms vary in menu costs and their frequency of price adjustment (Bhattarai and Schoenle, 2014; Anderson et al., 2015). Such *firm-side heterogeneity* of tracking and pricing technology has been relatively overlooked.

This paper aims to provide a unified theoretical foundation for various dispersed price patterns in dynamic oligopoly markets by incorporating both consumer-side and firm-side heterogeneity. The model can be regarded as a synthesis of the classic Varian (1980) model of sales and a textbook collusion model, where firms interact in an infinitely repeated pricing game and past prices are publicly observed (Athey et al., 2004). A fraction of consumers are *shoppers*, and a portion of firms are *trackers*. Shoppers and trackers can observe all prices in the market at zero cost, whereas non-shoppers (with positive search costs) and non-trackers (with positive menu costs) cannot.

Prices are chosen over an infinite time horizon.[2] Non-trackers behave similarly to the firms in the static Varian (1980) model: they commit to prices once and for all at $t = 0$. Trackers resemble the firms in a repeated pricing game: they simultaneously set prices each period $t = 1, 2, \ldots$. The innovation of our approach lies in the integration of such firms with different menu costs into a single Varian (1980) style model. For simplicity, we assume that non-trackers commit to prices without knowing how many of the rival firms

[1] There are also various tools that enable firms to track rivals' prices, receive price-drop alerts, and engage in automated repricing.

[2] In practice, the duration of the period for which non-tracker prices remain fixed could vary, ranging from brief instances such as an hour to more extended periods like a week, depending on the pricing technologies adopted by firms. A working paper version of this paper shows how our game can be converted to a setup where the time horizon remains finite.

they face are trackers while trackers immediately observe the number of trackers, which they could otherwise learn over time.

In practice, tracking and pricing can be delegated to autonomous price algorithms. Previous work stresses the potential of tracking technologies to facilitate collusion (Calvano et al., 2020; Klein, 2021; Johnson et al., 2023).[3] Our new approach highlights the significance of comprehending interactions between trackers and non-trackers. The dynamic model we study supports both collusive and non-collusive mixed equilibria, where the effects of tracking hinge on the prices chosen by non-trackers à la Varian (1980).

Section 3 begins the study by investigating duopoly pricing. As the prices set by non-trackers provide shoppers a viable outside option, we find that trackers derive lower payoffs in the presence of non-trackers, compared to collusion models involving only trackers. However, relative to the non-collusive benchmark provided by Varian (1980), where firms are non-trackers, consumer surplus diminishes. Namely, if a single tracker competes with a non-tracker, our prediction across all equilibria is unique. The tracker matches any price set by a non-tracker to capture shoppers. While such price matching does not affect the expected non-tracker payoff, it results in less elastic demand for these firms, encouraging non-trackers to charge higher prices. As a result, we observe that tracking increases market prices, transferring payoffs from consumers to trackers.

Section 4 considers more complex triopoly pricing. When several trackers compete in the market, pricing depends on their ability to collude. If collusion is impossible, two trackers adopt symmetric mixed strategies, setting new prices independently each period. Although changes prevent a tracker from learning its rival's price, the presence of a non-tracker in the market, with a fixed price, affects the pricing incentives for trackers. As trackers cannot compete for shoppers without matching the price of a non-tracker, they either mix beneath it, or set the monopoly price, but never use intermediate prices. This new price behavior leads to an atom and a gap in the equilibrium price distribution of two *trackers*. As the payoffs of two trackers remain as in Varian (1980), the non-collusive equilibrium represents the best-case scenario for consumers.

Under asymmetric collusion, two trackers jointly implement an asymmetric pricing strategy, which would be optimal for a merged unit owning both trackers. One tracker (leader) matches the price of a non-tracker to attract all shoppers, while the other one (follower) sets the monopoly price to extract maximal payoffs from its captives. This asymmetric scheme results in the highest joint payoffs for trackers across all market equilibria, defining the worst-case scenario for consumers. To share profits equally, firms could alternate between leading and following.

Under symmetric collusion, trackers employ identical mixed strategies, preventing the use of different price responses. Consequently, tracker optimal symmetric collusion re-

[3] This is exemplified by the EC's 2018 fine of 111 million EUR imposed on consumer electronics companies *Asus*, *Denon & Marantz*, *Pioneer*, and *Philips* for using advanced price monitoring algorithms.

quires both trackers to match the price of a non-tracker. As the payoffs from shoppers are shared, the average payoff for each tracker is lower than under asymmetric collusion. Two trackers are thus not willing to match every price set by a non-tracker, unlike a single tracker in the market is. Specifically, we identify a *collusion threshold* on the non-tracker price below which trackers prefer to forego symmetric collusion, yielding a lower tracker payoff than focusing on shoppers. When the non-tracker's price falls below this threshold, trackers revert to the non-collusive equilibrium, yielding payoffs equivalent to those in Varian (1980).

A notable finding of this paper is that there are equilibria where non-trackers price below the collusion threshold.[4] This mirrors limit-pricing since the non-trackers who offer these low prices face less competition from colluding trackers.[5] Limit-price strategies give rise to a gap in the equilibrium price distribution of *non-trackers*, right above the collusion threshold, because the prices beneath the threshold are more sheltered from competition than the higher prices. A single gap arises in a triopoly, but when the number of firms is higher, there could be several such gaps.

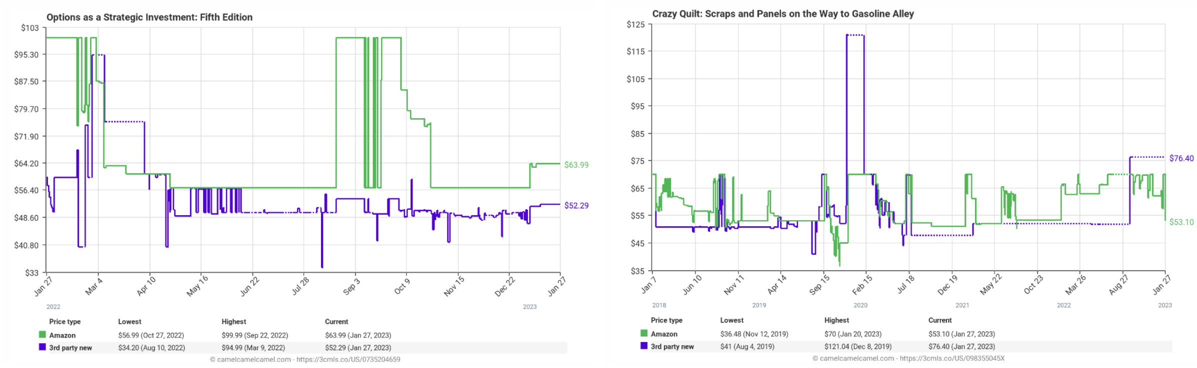


Figure 1: Best third-party (blue) and *Amazon* (green) prices at *Camelcamelcamel.com*.

Our new model turns out to be instrumental for reconciling salient features of pricing dynamics. Figure 1 depicts the price movements of two books available at *Amazon*, testifying to the richness of authentic price patterns in online markets.

1. *Price matching. Atomistic price.* The lowest third-party price quickly matches most price cuts by other firms – like the leading tracker. *Amazon* keeps its price unchanged for extended times while being undercut – as the following tracker.

2. *Random prices. Avoided prices. Amazon* either offers the same high fixed price or a much lower price but rarely the prices in between. Third-party seller's prices appear to be randomized and their price mixing mostly occurs below *Amazon*'s price.

[4]Multiple such equilibria arise since the collusion threshold converts tracker and non-tracker prices into *strategic substitutes.*

[5]The mechanism is different from the usual one (Milgrom and Roberts, 1982) because we do not study entry deterrence.

A closely aligned interpretation, as suggested by EC (2014), is that third-party sellers act as *price leaders*, while *Amazon* assumes the role of a *price follower*. Our model is agnostic about whether the highest prices are set by trackers or non-trackers. However, the data perhaps best fit a scenario in which third-party sellers include non-trackers (who commit to low prices) and trackers (who match those prices), while *Amazon* acts as a tracker, either setting prices below the non-trackers or offering the monopoly price. Consistent with our framework, which integrates a static sales model with a repeated pricing game, the mixed pricing pattern aligns with Varian (1980), while price rigidity may reflect the tacit collusion described by Green and Porter (1984).

By contrast, previous research does not provide any reason for why *Amazon* persistently prices much higher than its third-party rivals, while a monopoly price strategy could be the best-response of both a tracker and a non-tracker against a tracker in our model. In addition, earlier work cannot explain gaps in the equilibrium price distribution, which naturally arise in our model of heterogeneous price commitments. While atoms prevail in models of asymmetric firm prominence (Narasimhan, 1988; Wilson, 2010; Hämäläinen, 2018), gaps don't. Our main contribution lies in providing a rigorous theoretical account of such configurations that feature prominently in markets for homogeneous products.[6] We clarify the connections of our paper to the literature below.

Dynamic oligopoly pricing. Maskin and Tirole (1988) develop a theory of dynamic oligopoly pricing in an alternating offer Markov model with a finite pricing grid. Our paper also differs from their analysis by allowing for prolonged commitments to prices and consumer heterogeneity. Brown and MacKay (2023) explore a setting with competing price algorithms that either have different pricing frequencies or allow automatic price updating. They find that market prices lie between Bertrand and Stackelberg prices, highlighting the capacity of algorithms to raise prices without collusion. The paper abstracts from consumer heterogeneity, featured in our Varian (1980) style model.

Static oligopoly pricing. In classic oligopoly analyses (Butters, 1977; Varian, 1980; Burdett and Judd, 1983), price dispersion arises due to heterogeneously informed consumers. Even so, the persistence of equilibrium price dispersion in a dynamic setting where firms become aware of rival prices remains an open research question. Myatt and Ronayne (2019) develop a theory of stable equilibrium price dispersion, based on commitments to list prices that can only be discounted but not increased. In contrast to our analysis, their model results in just two pure equilibrium pricing strategies: the leading firm offers a discount price while the followers choose a monopoly-level price.

Collusion. Our work also relates to collusion with search and imperfect price information (Petrikaitė, 2016; Schultz, 2017; De Roos and Smirnov, 2020, 2021). Unlike most

[6] Naturally, many price patterns still remain unexplained by our purposely stylized model, which fails to account for, e.g., learning over product life-cycle (Bergemann and Välimäki, 2006), information costs of price adjustment (Alvarez et al., 2011), and strategic timing of purchase choices (Garrett, 2016).

prior work, we consider markets with equilibrium price dispersion, where collusive prices can remain below other prices. Empirical research on collusion by algorithms is focused on symmetric pricing technologies. Calvano et al. (2020, 2021) observe that price algorithms learn to charge collusive prices sustained by brief punishment phases. Assad et al. (2024) find that duopoly prices increase only if both firms adopt algorithmic pricing techniques. Cason and Friedman (2003) and Cason et al. (2021) provide laboratory evidence of price dispersion and price correlation in the Burdett and Judd (1983) noisy search model.

The article is organized as follows. The basic model is introduced in Section 2. Section 3 describes duopolistic equilibria while Section 4 considers equilibria for triopolies. Regulatory implications are laid out in Section 5. Section 6 closes with a brief discussion of model extensions. Proofs are delegated to the Appendix.

2 Model

We first study competition in a duopoly with (long-lived) firms $i, j = 1, 2$, catering to a unit mass of (short-lived) consumers in each period $t = 1, 2, \ldots$ Firms choose prices for identical products, and each consumer decides from which firm to buy a product. If a consumer purchases from firm i at price p^i, the utility of the consumer is $1 - p^i$. The monopoly price is unity. If firm i sells at price p^i to a mass $m^i(p^i)$ of consumers, the payoff that the firm derives is $m^i(p^i)p^i$.[7] Costs of production are normalized to zero. Firms discount future expected payoffs by a common discount factor $\delta < 1$.

Firms are uncertain regarding a consumer's information type. The probability that a consumer is a shopper is $\mu > 0$. A *shopper* observes all market prices when entering the market, whereas a *non-shopper* only observes one of the prevailing prices at random. Firms also differ in their technology types. The likelihood that a firm is a tracker is $\tau > 0$. A *tracker* sees past market prices and can subsequently reset its price. A *non-tracker* cannot observe the rival's price or, equivalently, cannot respond by changing its own. Whether a firm is a tracker is independent of whether its opponent is one.

To specify a tractable framework for analysis, we consider a simple infinite horizon model:

1. Period $t = 0$.

   (a) Nature draws firm types. A non-tracker only observes its own type while a tracker also observes its rival's type.

   (b) Non-trackers choose their prices simultaneously and are committed to charge

[7] The interpretation is that all products either remain homogeneous (e.g., a market for a certain book) or they have observable value of q^i and production costs of $c^i = q^i - 1$ (e.g., a market for branded ware).

these prices thereafter.[8] A tracker observes its rival's price.

2. Periods $t = 1, 2, \ldots$

   (a) Trackers choose their prices simultaneously and are committed to charge these prices in the current period. A tracker observes its rival's price.

   (b) A unit mass of consumers enters the market. Consumers make purchases at current prices and exit the market.

Shoppers purchase from the firm offering the lowest price, while non-shoppers choose randomly among the available prices. If a tracker and a non-tracker offer the same price, shoppers will buy from the tracker. Otherwise, ties are broken at random.

We focus on Perfect Bayesian Equilibria (PBE) because a firm's opponent could have different types. In a PBE, the strategies of firms are optimal given their beliefs. Equilibrium path beliefs are derived from the strategies by Bayesian updating.

The initial pricing choices of non-trackers at $t = 0$ take place under incomplete information. However, rival types become observable to trackers before they choose their prices at $t = 1, 2, \ldots$. Thereby, the subsequent price decisions of trackers are made under complete information.

The strategy of a non-tracker i is simply a choice of a fixed price $p_0^i \in [0, 1]$. The strategy of a tracker i maps a history $h_t = (p_k^i, p_k^j)_{k<t}$ of past market prices to a price choice $p_t^i \in [0, 1]$. For simplicity, time and firm indexes are omitted in the subsequent analysis when possible.

The goal of this paper is to characterize i. the equilibrium that maximizes the joint payoffs of trackers when δ is above a given collusion threshold and ii. the unique non-collusive equilibrium that persists for all values of δ, including δ close to zero. In this paper, we only investigate these two cases.

The former case represents the *worst equilibrium for consumers* in that it minimizes the expected payoffs of shoppers and non-shoppers among all equilibria, while the latter case defines the *best equilibrium for consumers*. Collusive tracker prices first-order stochastically dominate the non-collusive ones.

Benchmarks. When both firms are non-trackers, prices are chosen once and for all, but new short-lived consumers enter the market each period. In this case, our model essentially simplifies into the static Varian (1980) model.

Remark 1 (Non-collusive benchmark) *When $\tau = 0$ and $\mu \in [0, 1)$, a unique equilibrium arises for any value of δ where firms' randomized price strategies are supported over $[\frac{1-\mu}{1+\mu}, 1]$. A firm's expected per-period payoff is given by $\frac{1-\mu}{2}$.*

[8] The actual duration of firms' price commitments could be as short as a minute or an hour, depending on the prevailing price technology. A working paper version of this paper provides an example model.

When both firms are trackers, our model defines a repeated game, with the stage game given by Varian (1980). For the lowest values of δ, the unique equilibrium remains an independently repeated version of Varian (1980). For high enough values of δ, however, collusive equilibria exist, with higher per-period payoffs, $\pi^i > \frac{1-\mu}{2}$.[9]

In symmetric collusion, trackers' strategies and payoffs are identical, $\pi^i = \pi^j \leq \frac{1}{2}$. In asymmetric collusion, one tracker could target shoppers while the other one focuses on non-shoppers. As these roles can be changed by period, any average payoffs where $\pi^i + \pi^j \leq 1$ and $\pi^i > \frac{1-\mu}{2}$ can be attained for a sufficiently high δ.

Remark 2 (Collusive benchmarks) *When $\tau = 1$ and $\mu \in (0, 1]$, there exists a continuum of equilibria for any sufficiently high δ where a firm's per-period payoff range is $(\frac{1-\mu}{2}, \frac{1}{2}]$ for symmetric collusion ($(\frac{1-\mu}{2}, 1 - \frac{1-\mu}{2})$ for asymmetric collusion).*

3 Duopoly equilibrium

To solve the model from the back to the beginning, we start by considering a game where a tracker plays against a tracker in Section 3.1. We proceed in Section 3.2 to a tracker's optimal (pure) strategy against a non-tracker, and derive in Section 3.3 a non-tracker's optimal (mixed) strategy against an unknown opponent. The equilibrium price distribution and the payoff of a tracker is derived in Section 3.4, and consumer surplus and the division of equilibrium payoffs investigated further in Section 3.5.

3.1 Tracker's problem against tracker

The *tracker-optimal* equilibrium of the subgame where a tracker meets a tracker features collusive strategies if δ is high. Tracker prices center around the monopoly price unity. Collusion can be sustained by the threat that, if either firm deviates from its collusive price strategy, the other one reverts to the *consumer-optimal* equilibrium. This punishment equilibrium is an independently repeated version of the Varian (1980) stage game where tracker prices are mixed and the per-period tracker payoff is $\frac{1-\mu}{2}$.

The equilibrium of a tracker-tracker subgame at history h_t is given by the maximizers (p^i, p^j) of

$$\Pi^i(h_t) = \max_{p^i} \pi(p^i, p^j) + \delta E \Pi^i(h_{t+1}(p^i, p^j)), \tag{1}$$

where the value function of firm i is denoted by Π^i and the per-period firm payoff is

[9] Collusion can be sustained by reverting to the non-collusive equilibrium upon a deviation. Lower payoffs for firms require a higher discount factor to discipline deviations from collusion.

defined by

$$\pi(p^i, p^j) = \begin{cases} \frac{1-\mu}{2} p^i, & \text{for } p^i > p^j, \\ \frac{1}{2} p^i, & \text{for } p^i = p^j, \\ \frac{1+\mu}{2} p^i, & \text{for } p^i < p^j. \end{cases}$$

In each period, a firm's demand varies between $\frac{1-\mu}{2}$ (only captives) and $\frac{1+\mu}{2}$ (also shoppers). In other words, a firm obtains *at least* the payoff of $\frac{1-\mu}{2}$ by setting the monopoly price 1 and *at most* $\frac{1+\mu}{2} p^i$ by setting any lower price p^i. Any feasible equilibrium payoffs can thus be attained without requiring that a firm offers a price below the low price bound $\underline{p}^1 = \frac{1-\mu}{1+\mu}$, as defined by

$$\frac{1+\mu}{2} \underline{p}^1 = \frac{1-\mu}{2} 1.$$

In general, we find that any equilibrium where tracker payoffs exceed the payoffs in the consumer-optimal repeated Varian (1980) model can be sustained in the subgame between two trackers; as standard for collusion models, we thus obtain the folk theorem. Proposition 1 outlines the consumer-optimal equilibrium and Proposition 2 defines the tracker-optimal equilibrium.

Proposition 1 *There exists a non-collusive equilibrium where tracker prices are mixed independently across time periods following the probability distribution G, given by*

$$G(p) = 1 - \frac{1-\mu}{2\mu} \frac{1-p}{p}. \tag{2}$$

This equilibrium is consumer-optimal and generates the lowest tracker payoffs among all equilibria, with the expected present discounted value of tracker payoffs being $\frac{1-\mu}{2(1-\delta)}$.

Proposition 2 *For any $\delta \geq \frac{1}{2}$, there exists a collusive equilibrium where tracker prices are fixed at unity. This equilibrium is tracker-optimal and generates the lowest consumer payoffs among all equilibria. The expected present discounted value of tracker payoffs is $\frac{1}{2(1-\delta)}$.*

Focusing on collusive equilibria is warranted, as Calvano et al. (2020) show that automated price algorithms that track rival prices easily learn to charge collusive prices. The worst-case scenario for consumers is when trackers collude at the monopoly price.[10]

Our subsequent analysis reveals that interactions between trackers and non-trackers also notably influence payoffs. The presence of trackers in the market drives up prices for non-trackers, while the presence of non-trackers leads to price reductions for trackers.

[10]This tracker-optimal equilibrium represents the equilibrium that trackers would be most willing to play, akin to the *undefeated equilibrium* (Mailath et al., 1993) in the refinement literature.

3.2 Tracker's problem against non-tracker

A tracker's problem against a non-tracker is simpler than its problem in the previous tracker-tracker game. Because ties are broken in the favor of the tracker and the non-tracker is unable to retaliate after a price cut, a tracker will match any price of a non-tracker that lies above the low price bound of $\underline{p}^1$ to capture shoppers.

On one hand, by choosing the monopoly price $p^i = 1$ while its rival offers a price $p^j < 1$, a tracker secures the expected per-period payoff of

$$\pi^1 = \frac{1-\mu}{2}$$

from the non-shoppers who observe its price but fail to observe the rival price. On the other, by matching its opponent's price, a tracker obtains

$$\pi^1 = \left(\frac{1-\mu}{2} + \mu\right) p^j$$

by selling to its captives and to shoppers who observe the rival price p^j along with its tie-breaking price offer of $p^i = p^j$.

By comparing the expected payoffs of a tracker, we conclude that a tracker weakly benefits from matching the price of a non-tracker if $p^j \geq \underline{p}^1$. Otherwise, a tracker rather charges the monopoly price.

Because a non-tracker commits to its price at $t = 0$, neither a tracker has a reason to change it price. The prices of a tracker and a non-tracker thus remain constant after $t = 1$, leading to stable prices.[11]

3.3 Non-tracker's problem

A non-tracker chooses its price without knowing whether it encounters a tracker or another non-tracker. Therefore, a non-tracker does not know if its price will be matched by a rival while the non-tracker's price stays fixed. Similarly to Varian (1980), the prospect of facing another non-tracker makes it optimal for a non-tracker to price in mixed strategies.

We derive the unique symmetric duopoly equilibrium. Knowing that a firm never offers prices below $\underline{p}^1$,[12] the per-period payoff to a non-tracker can be written as

$$\pi^0(p^i, F) = \left(\frac{1-\mu}{2} + \mu(1-\tau)(1-F(p^i)) + \frac{\mu(1-\tau)}{2}\mathbf{1}(p^i = p^j)\right) p^i, \tag{3}$$

because a non-tracker only wins shoppers when it counters another non-tracker whose

[11] Tracker-optimal collusive prices in a tracker-tracker game similarly remain constant after $t = 1$.

[12] Any price $p^i < \underline{p}^1$ gives a firm at most $\frac{1+\mu}{2(1-\delta)} p^i < \frac{1+\mu}{2(1-\delta)} \underline{p}^1$, while $p^i = 1$ gives $\frac{1-\mu}{2(1-\delta)} = \frac{1+\mu}{2(1-\delta)} \underline{p}^1$.

prices follow the distribution F (F denotes the cdf and f the related pdf; $\mathbf{1}(p^i = p^j)$ indicates a potential tie with the rival's price p^j).

In the unique symmetric duopoly equilibrium, a non-tracker's per-period payoff is given by

$$\pi^0(p^i, F) = \left(\frac{1-\mu}{2} + \mu(1-\tau)(1-F(p^i))\right)p^i, \tag{4}$$

because a non-tracker only wins shoppers if the rival is a non-tracker and has a higher price offer.

Compared to the usual Varian (1980) model where all firms are non-trackers, we thus find that the demand of a non-tracker from shoppers becomes less elastic. Next, we show that less elastic demand implies that non-trackers set higher average prices with the (potential) presence of trackers.

Following the standard logic from Varian (1980) with the updated demand function, we find that a non-tracker's mixed price strategy can be expressed as

$$F(p) = 1 - \frac{1-\mu}{2(1-\tau)\mu}\frac{1-p}{p}, \tag{5}$$

corresponding with the expected per-period payoff of $\pi^0 = \frac{1-\mu}{2}$ and the expected non-tracker's price of[13]

$$E(p^0) = \frac{1-\mu}{2(1-\tau)\mu}\ln\left(1 + \frac{2(1-\tau)\mu}{1-\mu}\right). \tag{6}$$

The expected price increases in τ. The lowest equilibrium prices $\underline{p}^0$ of non-trackers can be solved as follows

$$\pi^0 = \left(\frac{1-\mu}{2} + \mu(1-\tau)\right)\underline{p}^0 \Longrightarrow \underline{p}^0 = \frac{1-\mu}{2(1-\tau)\mu + 1 - \mu} > \underline{p}^1 = \frac{1-\mu}{1+\mu}. \tag{7}$$

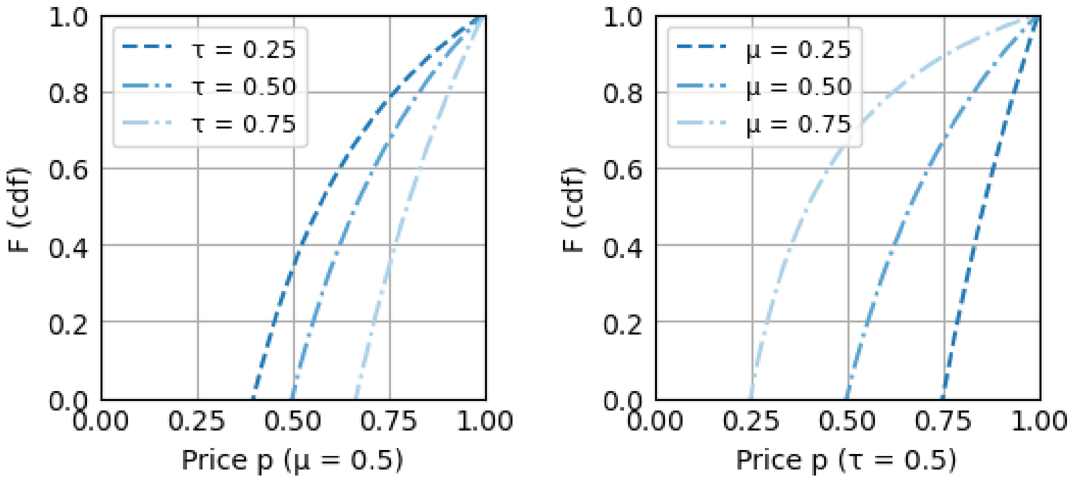
Relative to the Varian (1980) model in which $\tau = 0$ we can thus see that tracker presence in the market ($\tau > 0$) narrows the support of the equilibrium price distribution from the standard $[\underline{p}^1, 1]$ to $[\underline{p}^0, 1]$. The entire equilibrium price distribution of non-tracker prices F also becomes higher, in terms of first-order stochastic dominance.

We notice further that, while a non-tracker's expected price rises with more trackers, its expected payoff remains constant as the number of trackers increases. The broader presence of trackers in a market thus shifts payoffs from consumers to trackers but does not alter the payoff division between consumers and non-trackers.

Comparative statics of equilibrium prices F are simple to derive by Eq. (5), which shows that $\frac{\partial F}{\partial \tau} < 0$ and $\frac{\partial F}{\partial \mu} > 0$. Figures 2a and 2b depict F for different μ and τ. Figure

[13] Note for later use that the cdf of the minimum of two non-tracker prices is $F^{(1)}(p) = 1-(1-F(p))^2 = 1 - \frac{1}{4\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(1-\frac{1}{p}\right)^2$ and the pdf $f^{(1)}(p) = 2(1-F(p))f(p) = -\frac{1}{2\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(1-\frac{1}{p}\right)\frac{1}{p^2}$, which gives the average minimum price $E(p^{(1)}) = \frac{1}{2\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(\ln \underline{p}^0 + \frac{1}{\underline{p}^0} - 1\right)$.

2a confirms our finding that a higher number of trackers raises non-tracker prices, while Figure 2b demonstrates a qualitatively similar price effect when there are fewer shoppers. As seen from (5), prices are generally more sensitive to the fraction of shoppers than trackers.



(a) $F(p)$ (b) $F(p)$

Figure 2: Equilibrium price distribution of non-trackers.

3.4 Tracker payoff

We proceed to derive the equilibrium price distribution and the payoff of a tracker. As described in Sections 3.1 and 3.2, tracker behavior differs with a tracker and a non-tracker. Multiple equilibria found in the tracker-tracker game lead to different average prices for trackers. We cover the consumer-optimal non-collusive case and the tracker-optimal collusive case separately.

3.4.1 Best-case scenario for consumers

Presuming that two trackers implement the non-collusive equilibrium of Proposition 1 represents the best-case scenario for consumers. If the rival is another tracker, a tracker chooses a mixed price from the classic Varian (1980) distribution G in (2). Instead, if the opponent is a non-tracker, a tracker matches the non-tracker's price, which follows the newly derived distribution F in (5).

As a result, the unconditional distribution of tracker prices H becomes a mixture $H(p) = \tau G(p) + (1-\tau)F(p)$ of two continuous discount distributions G and F. The strategy yields a tracker the expected per-period payoff of

$$\pi^1 = \tau\frac{1-\mu}{2} + (1-\tau)\frac{1+\mu}{2}E(p^0) = \tau\frac{1-\mu}{2} + \frac{1-\mu^2}{4\mu}\ln\left(1+\frac{2(1-\tau)\mu}{1-\mu}\right).$$

This payoff exceeds the standard payoff $\frac{1-\mu}{2}$ of a non-tracker in Varian (1980) for two reasons. First, when a tracker competes against a non-tracker, it obtains shoppers with certainty by matching the non-tracker's price. Second, because non-tracker prices are higher in a market with trackers, the average price at which shoppers buy exceeds that in Varian (1980).

When a tracker faces a non-tracker, its ability to capture shoppers by matching the non-tracker's price means that the tracker obtains a higher payoff, even though both a tracker's and a non-tracker's prices follow an identical distribution function, F. A non-tracker only wins shoppers with a positive probability, $1 - F(p)$, when it competes with another non-tracker.

3.4.2 Worst-case scenario for consumers

Supposing that two trackers coordinate to the collusive equilibrium of Proposition 2 represents the worst-case scenario for consumers. If the rival is a non-tracker, tracker prices follow F and, if the opponent is a tracker, the tracker price equals unity – two trackers collude at the highest possible price.

The unconditional distribution of tracker prices J becomes a mixture $J(p) = (1 - \tau)F(p) + \tau\mathbf{1}\{p \geq 1\}(p)$ of a continuous price distribution F and an indicator (step) function $\mathbf{1}\{p \geq 1\}$, representing a mass point at price one. The expected tracker payoff under this pricing strategy can be expressed as

$$\pi^1 = \tau\frac{1}{2} + (1-\tau)\frac{1+\mu}{2}E(p^0) = \tau\frac{1}{2} + \frac{1-\mu^2}{4\mu}\ln\left(1 + \frac{2(1-\tau)\mu}{1-\mu}\right).$$

Apart from the reasons laid out in Section 3.4.1, this tracker payoff exceeds the standard payoff $\frac{1-\mu}{2}$ of a non-tracker in Varian (1980) because two trackers coordinate to the monopoly price and divide the monopoly payoff.

In summary, pricing changes significantly with the presence of trackers. We observe that tracking harms consumers, irrespective of whether trackers collude. The expected market prices of trackers and non-trackers exceed the prices in Varian (1980) because a tracker will match the price that a non-tracker offers, leading to higher prices in a mixed market where both are present.

3.5 Consumer surplus

3.5.1 Best-case scenario for consumers

In the best-case scenario for consumers, the consumer surplus is given by $\mu = 1 - \frac{1-\mu}{2} - \frac{1-\mu}{2}$ with two trackers. This matches the consumer surplus in the Varian (1980) model of sales. By contrast, in a market with either a tracker and a non-tracker or two non-trackers, the expected consumer surplus is less than μ. This implies that tracking harms consumers even in the best-case scenario.

Specifically, the expected consumer surplus for a single cohort of consumers can be expressed as

$$CS = \tau^2\mu + 2\tau(1-\tau)(1-E(p^0)) + (1-\tau)^2(\mu(1-E(p^{(1)})) + (1-\mu)(1-E(p^0))),$$

where $E(p^0)$ gives the expected price of a non-tracker and $E(p^{(1)})$ the minimum of non-tracker prices.

The logic behind this result is as follows. In a market with a tracker and a non-tracker, shoppers and non-shoppers pay the same average price $E(p^0)$ since the prices of a tracker and a non-tracker are identical. This reduces consumer surplus because shoppers do not benefit from their ability to purchase at the lowest market prices.

In a market with two non-trackers, shoppers obtain the minimum price $E(p^{(1)})$ and non-shoppers the average price $E(p^0)$ as usual. These prices are higher than their counterparts in Varian (1980) as non-trackers face less elastic demand. Therefore, the decomposition of consumer surplus into that of shoppers and non-shoppers

$$\mu(1-E(p^{(1)})) + (1-\mu)(1-E(p^0)),$$

falls short of the standard consumer surplus of μ.

3.5.2 Worst-case scenario for consumers

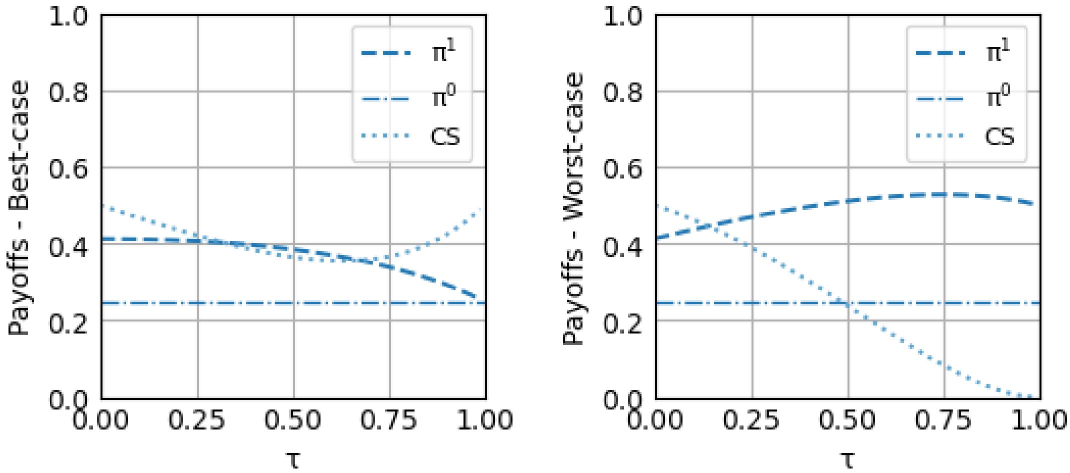
In the worst-case scenario for consumers, consumer surplus is zero with two trackers, who collude at the monopoly price unity. Total consumer surplus is hence given by

$$CS = \left(2\tau(1-\tau) + (1-\tau)^2(1-\mu)\right)\left(1-E(p^0)\right) + (1-\tau)^2\mu\left(1-E(p^{(1)})\right), \tag{8}$$

since all consumers pay $E(p^0)$ in a market with a non-tracker and a tracker. With two non-trackers, non-shoppers are charged $E(p^0)$ while shoppers are paying $E(p^{(1)})$.

Non-shoppers and shoppers alike thus face higher prices in markets where trackers are cartellized or tacitly collude. The consumer surplus is hence lower in this worst-case scenario for consumers than in the best-case scenario, where trackers compete à la Varian (1980). While the focus of the paper is on the best and worst equilibria for consumers, the ranking of collusive and non-collusive equilibria for consumer surplus does not depend on whether two trackers collude at the monopoly price or any collusive prices, yielding both trackers a higher payoff than the non-collusive Varian (1980) equilibrium.

Intuitively, the presence of trackers in a duopoly market always reduces consumer surplus since trackers receive a higher payoff while the payoff of a non-tracker remains



(a) $\mu = 0.5$ (b) $\mu = 0.5$

Figure 3: Payoffs in the best-case and worst-case scenarios.

unchanged. Figure 3 illustrates the impact of trackers on payoffs in the best-case (a) and worst-case (b) scenarios for consumers. Note that payoffs do not add up to one unless $\tau = \frac{1}{2}$. This is because, when τ elevates, the average number of trackers receiving π^1 increases whereas that of non-trackers obtaining π^0 decreases.

Best-case for consumers. Figure 3a depicts that, in the absence of collusion, consumer surplus is non-monotone in τ. On one hand, the expected price $E(p^0)$ charged in a market with non-trackers is increasing in τ. On the other, a tracker is more likely to meet another tracker, reducing the expected price offered by a tracker. Recall that Varian (1980) provides a valid estimate of firm payoffs when most firms are non-trackers or trackers in the absence of collusion. Thus, only the intermediate value range of τ negatively affects consumers. This result means that firms benefit the most when they expect to be different in their ability to reset prices. The expected asymmetry implies that firms expect to focus on different subsets of consumers, benefiting both firms.

Worst-case for consumers. When trackers collude at the monopoly price, the expected consumer surplus continuously transitions from that predicted by Varian (1980) at $\tau = 0$ to the one given by a textbook collusion model at $\tau = 1$. Nevertheless, interestingly, we find that trackers only benefit from the presence of other trackers up to a certain point, after which their payoffs decline as the number of colluding trackers increases in the market. As described in Figure 3b, a tracker's per-period payoff π^1 is therefore non-monotone in τ.

The mechanism is novel to the literature. There is a price effect and a demand effect. When a tracker colludes with another tracker, it obtains a higher collusive price of 1 but captures only half the shoppers. Conversely, if a tracker competes with a non-tracker, it receives a lower expected price of $E(p^0)$ but attracts all the shoppers.

A non-tracker offers a higher mean price when its competitor is more likely to be a tracker. Therefore, the difference between the collusive monopoly price of 1 and the expected non-tracker price $E(p^0)$ dissipates as $\tau \to 1$. For high enough values of τ, the negative demand effect of collusion thus dominates the positive price effect.

4 Triopoly equilibrium

More elaborate price patterns arise in a larger market where firms differ in commitment power and consumers in price information. To analyse such markets with $m > 2$ firms, we assume, as before, that all non-trackers commit to prices once and for all at time $t = 0$ whereas all trackers can simultaneously set new prices at $t = 1, 2, \ldots$ We concentrate next on triopoly markets, which suffice to show the main insights.[14]

We restrict our analysis to symmetric strategies for non-trackers. With more than two firms, a continuum of asymmetric equilibria otherwise arises, as shown by Baye et al. (1992). Collusive tracker strategies can be asymmetric (Section 4.1) or symmetric (Section 4.2). Symmetric collusion is the next-best outcome for trackers if they fail to implement the asymmetric scheme, maximizing the joint payoffs of trackers.

4.1 Asymmetric tracker collusion

For any given price p^0, asymmetric collusion provides the highest tracker payoffs, conversely defining the worst-case scenario for consumers. Asymmetric collusion can therefore be defended as the equilibrium preferred by trackers who coordinate their strategies; the idea dates back to Mailath et al. (1993). At the same time, while asymmetric collusion maximizes the average tracker payoff, it demands more collaboration from trackers. The difficulty in maintaining asymmetric collusion, whether done tacitly or through an illegal cartel, may lead firms to favor simpler symmetric strategies, studied in Section 4.2.

In both a duopoly and a triopoly, trackers who compete in the absence of non-trackers derive the highest joint payoffs by colluding at the monopoly price. By contrast, if there are two trackers and a non-tracker in the market, maximal tracker payoffs require asymmetric strategies where ii. one tracker (leader) matches the non-tracker price to capture all shoppers while i. the other one (follower) offers the monopoly price, thus forgoing shoppers. Under asymmetric collusion, the trackers thus assume different pricing strategies to obtain the same profit that a single tracker with two price instruments derives.[15]

Non-trackers apply mixed price strategies, resembling the ones they use in a duopoly. However, because non-trackers only attract shoppers if they solely face non-trackers, their prices are higher in a triopoly than in a duopoly for any τ.

[14] We have studied also markets with more firms, which allow for more complex price patters, e.g., one gap per each collusion threshold c_r and several payoff-ranked equilibria.

[15] For a multi-price monopolist, the price strategy would resemble the *hi-lo* strategy described in Hämäläinen (2022) in the context of multiproduct duopoly competition.

Proposition 3 *Tracker-optimal collusive equilibrium is asymmetric. If there is one tracker, it matches the lowest non-tracker price. If there are two trackers, one matches the price of a non-tracker and the other charges the monopoly price. Non-tracker prices follow*

$$K(p) = 1 - \frac{1-\mu}{3\mu(1-\tau)^2}\frac{1-p}{p}.$$

In this worst-case scenario for consumers, total consumer surplus can be expressed as

$$\begin{aligned}
CS =&\tau^3(1-1)+\\
&3\tau^2(1-\tau)\left(\frac{1-\mu}{3}(1-1)+(\mu+2\frac{1-\mu}{3})\left(1-E(p^0)\right)\right)+\\
&3\tau(1-\tau)^2\left(2\frac{1-\mu}{3}\left(1-E(p^0)\right)+(\mu+\frac{1-\mu}{3})\left(1-E(p^{(1(2))})\right)\right)+\\
&(1-\tau)^3\left((1-\mu)\left(1-E(p^0)\right)+\mu\left(1-E(p^{(1(3))})\right)\right),
\end{aligned}$$

where $E(p^0)$ denotes the average price and $E(p^{(1(n))})$ the first (lowest) order statistic among n prices independently distributed according to K.

The asymmetric collusive scheme can be implemented in various ways to split payoffs, including by rotating which firm is the leader and which one the follower in each period. Because the non-tracker price is fixed, each period yields $(\mu + \frac{1-\mu}{3})E(p^0)$ to the leader and $\frac{1-\mu}{3}$ to the follower. An equal division of payoffs is thus attained by letting firm $i = 1$ ($i = 2$) lead in odd (even) periods and follow in even (odd) periods.

In reality, there is scant evidence of this kind of demand rotation in digital markets. More commonly, an asymmetric scheme is observed where the most agile firm in the market always underprices more complacent firms – including *Amazon* in Figure 1. The following trackers gain nothing from tracking. Asymmetric tracker collusion can thus explain why some firms may not invest in tracking technologies.

Asymmetric collusion features a novel dispersed price pattern in which trackers offer both the lowest and highest market prices. Perhaps surprisingly, the highest prices signify *low* current payoffs for the tracker who only caters to captive customers, while the lowest prices indicate *high* current payoffs for the tracker who captures all shoppers. In a mixed market, we should thus expect to see more varied collusive price behavior than a textbook model would predict.

4.2 Symmetric tracker strategies

We turn to investigate symmetric tracker strategies, which necessitate less coordination from the firms. Under asymmetric collusion, price patterns are closely aligned in a duopoly and triopoly (and generally, any oligopoly); if trackers and non-trackers are present in the market at the same time, the lowest price among the non-trackers is always matched by one of the trackers.

Under symmetric tracker strategies, the benefits of collusion are more limited in a triopoly, which leads to more complex pricing. To characterize symmetric equilibria, we start by defining the consumer-optimal non-collusive benchmark equilibrium in Section 4.2.1, which serves as the threat point supporting the tracker-optimal symmetric equilibrium, described in Section 4.2.2.

4.2.1 Consumer-optimal symmetric equilibrium

Pricing depends on the number of trackers in the market. If the market is populated by three trackers (who set prices at the same time) the unique symmetric equilibrium of the stage game is independently repeated across periods. The trackers thus derive the same payoffs as in Varian (1980). In contrast, if the market has one tracker and two non-trackers, the tracker (whose price is set the latest) always matches the lowest price among the non-trackers (who set their prices first). To complete the analysis, Proposition 4 delineates the remaining case of two trackers and one non-tracker in the market, where differences arise. Proposition 5 defines equilibrium pricing for a non-tracker.

Proposition 4 *Consider a triopoly with one non-tracker whose price is p^0. The equilibrium of the one-shot-game between two trackers is a mixed strategy L supported on $[\underline{p}^1, p^0] \cup \{1\}$, resulting in the tracker payoff of $\frac{1-\mu}{3}$. The probability $1 - \alpha(p^0)^2$ that a tracker matches p^0 is defined by the size of the mass point at unity, given by*

$$\alpha(p^0) = 1 - L(p^0) = \frac{1-\mu}{3\mu}\frac{1-p^0}{p^0}.$$

Remark 3 (Comparative statics) *The atom α is decreasing in μ and p^0.*

Proposition 5 *Consider a triopoly where two trackers choose prices from L. The equilibrium of the one-shot-game for non-trackers is a mixed strategy K supported on $[\underline{p}^0, 1]$ with $\underline{p}^0 > \underline{p}^1$, resulting in the non-tracker payoff of $\frac{1-\mu}{3}$. The distribution of non-tracker prices K depends on the probability $\alpha(p^0)$ that a tracker prices over p^0*[16]

$$K(p^0) = 1 - \frac{1}{1-\tau}\sqrt{\alpha(p^0) - \tau^2\alpha(p^0)^2}.$$

In a market with two trackers and a non-tracker, the trackers thus randomize between offering the monopoly price and a random price beneath the non-tracker. The non-tracker chooses a mixed price whose distribution first-order stochastically dominates the

[16] The corresponding Varian distribution would be $K(p^0) = 1 - \sqrt{\alpha(p^0)} = 1 - \sqrt{\frac{1-\mu}{3\mu}\frac{1-p^0}{p^0}}$.

standard Varian (1980) distribution. The resulting novel price dynamics showcase well how interactions between trackers and non-trackers enrich standard price strategies that would arise in a market with only trackers or non-trackers:

The effect of non-trackers on trackers. Trackers' payoffs remain as in Varian (1980). However, we find that their equilibrium price distribution has a novel gap $(p^0, 1)$ above the price p^0. The gap emerges since a tracker never sells to shoppers by setting a price above p^0. Therefore, there is no competition for shoppers between trackers at $p > p^0$ and no incentive to mix prices with rival trackers above this threshold. At the same time, the certainty of attracting captives encourages trackers to increase their prices above p^0, resulting in a mass point at unity.

The effect of trackers on non-trackers. Because trackers are less likely to match the price of a non-tracker in a triopoly than in a duopoly, non-trackers respond to reduced competition by offering higher expected prices than in duopoly markets. Simultaneously, competition intensifies among rival non-trackers, exerting downward pressure on their prices. Consequently, we find that, depending on the probability that a non-tracker faces two trackers versus two non-trackers, non-tracker prices can increase or decrease in a triopoly compared to a duopoly.

Propositions 4 and 5 describe the consumer-optimal equilibrium. As the payoff of one tracker with two non-trackers exceeds the payoff of a firm in Varian (1980), we can generally conclude that the presence of trackers harms consumers also in a triopoly.

The lowest equilibrium prices are given by

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu\right) \underline{p}^1 \Longrightarrow \underline{p}^1 = \frac{1-\mu}{1+2\mu}, \tag{9}$$

for trackers, and

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu\left((1-\tau)^2 + \tau^2\alpha^2(\underline{p}^0)\right)\right) \underline{p}^0 \tag{10}$$

for non-trackers. The intuition is that a tracker wins against any number of non-trackers when it chooses the lowest price $\underline{p}^1$. Instead, a non-tracker who chooses $\underline{p}^0$ only wins against two non-trackers or two trackers who set the monopoly price, which they do with the probability of $\alpha^2(\underline{p}^0)$. Because non-trackers derive less demand from shoppers, their lowest prices must lie below those of trackers: $\underline{p}^0 > \underline{p}^1$.[17]

4.2.2 Tracker-optimal symmetric equilibrium

While a single tracker is willing to match any price of a non-tracker, two trackers may prefer not to under symmetric tracker collusion. Specifically, we identify a *collusion threshold* on the non-tracker price below which symmetric tracker collusion becomes unprofitable.

[17] Price $\underline{p}^0$ can be uniquely defined as the highest solution in $(0, 1)$ of the quadratic equation (10).

This collusion threshold, $c = \frac{1-\mu}{1+\mu/2}$, is defined by the equation:

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \frac{\mu}{2}\right) c, \tag{11}$$

where the left-hand side (lhs) represents the maximal payoff under symmetric collusion at $p^0 = c$, equating it with the maximal payoffs achievable at higher prices $p > p^0$ on the right-hand side (rhs). If a non-tracker sets a price below the collusion threshold, a tracker unilaterally benefits from deviating from collusion. The strategy of charging the monopoly price and only selling to its captive segment then yields a higher payoff than matching the non-tracker's price alongside another tracker, which entails the trackers sharing the shopper market.

However, both trackers cannot set the monopoly price in the subgame where two trackers choose prices after observing that a non-tracker has selected the price $p^0 < c$. Otherwise, a tracker would benefit from matching the non-tracker price while undercutting the rival tracker's price; the rival could not punish.[18] In equilibrium, non-tracker prices below the collusion threshold therefore cause trackers to abandon symmetric collusion for the non-collusive equilibrium described by Proposition 4, where tracker payoffs align with those of Varian (1980).

Proposition 6 *Consider a triopoly with one non-tracker whose price is p^0. Assume high enough δ to sustain symmetric collusion. The tracker-optimal symmetric equilibrium is as follows: If $p^0 > c$, each tracker offers the collusive fixed price of p^0. If $p^0 \leq c$, each tracker chooses prices from L independently across time periods.*

Proposition 7 *In the tracker-optimal symmetric equilibrium, non-trackers follow a mixed strategy M: If $\tau \leq 1 - \frac{1}{\sqrt{2}}$, M is supported on $[\underline{p}^0, c] \cup [c+g, 1]$ and piece-wise defined as follows*

$$M(p^0) = \begin{cases} 1 - \frac{1}{1-\tau}\sqrt{\alpha(p^0) - \tau^2 \alpha(p^0)^2}, & \textit{for } p \leq c, \\ 1 - \frac{1}{1-\tau}\sqrt{\alpha(p^0)}, & \textit{for } p > c+g. \end{cases}$$

If $\tau > 1 - \frac{1}{\sqrt{2}}$, M is supported on $[\underline{\underline{p}}^0, 1]$ and defined as follows

$$M(p^0) = 1 - \frac{1}{1-\tau}\sqrt{\alpha(p^0)},$$

where the lower bound on tracker prices is $\underline{\underline{p}}^0 = \frac{1-\mu}{1+(3(1-\tau)^2-1)\mu} > c$.

We find that the equilibrium distribution M has a gap $(c, c+g)$ when trackers offer prices below the collusion threshold, for $\underline{p}^0 < c$. The gap arises since the prices below

[18] Our notion of symmetric collusion precludes the use of a randomization device determining which firm will match p^0. For a general theory of penal codes in repeated games, see Abreu (1988).

c are never met by two colluding trackers, unlike higher prices. By pricing beneath the collusion threshold at $c - \epsilon$ ($\epsilon \to 0$), a non-tracker obtains a higher payoff of

$$\left(\frac{1-\mu}{3} + \mu\tau^2\alpha(c)^2 + \mu(1-\tau)^2\left(1 - M(c)\right)^2\right)c,$$

whereas by setting a slightly higher price $c + \epsilon$ ($\epsilon \to 0$), a non-tracker receives the smaller payoff of

$$\left(\frac{1-\mu}{3} + \mu(1-\tau)^2\left(1 - M(c)\right)^2\right)c.$$

Therefore, a profitable deviation from above c to below it exists for any continuous distribution M.

Remark 4 (Comparative statics) *The gap g is increasing in τ.*

In a larger market with more than three firms, there could be several collusion thresholds, $c_r = \frac{1-\mu}{\frac{n}{r}\mu+1-\mu}$, c_2 for two trackers, c_3 for three trackers, etc. At the collusion threshold c_r, the residual surplus from shoppers $\mu p^{(1(n-r))}$, that r trackers would share equally, is too low to sustain symmetric collusion.

Collusion thresholds c_r are increasing in the number of trackers in the market r. As standard, collusion is therefore more difficult to sustain among more firms. Existence of multiple collusion thresholds also entails that there could be several gaps in the equilibrium price distribution of non-trackers because pricing right above a collusion threshold is dominated by pricing slightly below one.

When is symmetric collusion sustained? We presume that a deviation from collusion by one firm results in the other one reverting the mixed equilibrium determined by Propositions 4 and 5. The instantaneous benefit to a tracker from undercutting the rival is hence surpassed by the reduced future payoff if

$$\left(\mu + \frac{1-\mu}{3}\right)p^0 + \delta\frac{1}{1-\delta}\frac{1-\mu}{3} \leq \frac{1}{1-\delta}\left(\frac{\mu}{2} + \frac{1-\mu}{3}\right)p^0.$$

Remark 5 *Symmetric collusion is sustainable if $p^0 > c$ and $\delta \geq \frac{\frac{\mu}{3}p^0}{\left(\left(\mu+\frac{1-\mu}{3}\right)p^0 - \frac{1-\mu}{3}\right)}$.*

The existence of alternative lowest prices arises for $1 - \frac{1}{\sqrt{2}} < \tau \leq \frac{4-\sqrt{6}}{5}$ because the standard logic of *strategic complements* breaks down due to the collusion threshold. Competition with trackers is more intense above than below the threshold. In consequence, non-tracker and tracker prices turn into *strategic substitutes*: By offering a price below c, a non-tracker can force two trackers to price above it with positive probability, whereas offering a price above c_2 would cause the trackers to match its price.

The first pricing strategy is akin to a limit-price strategy in conventional entry deterrence models, such as Milgrom and Roberts (1982), because setting a price below c allows a non-tracker to limit its exposure to tracking. The mechanism is different from accounts where collusion is made unstable by firm-side heterogeneity, see Clark and Houde (2013) for network variation and Fonseca and Normann (2008) for asymmetric capacities.[19] Here, colluding trackers are homogeneous and non-trackers disturb collusion.

Overall, our triopoly analysis echoes our finding for a duopoly that i. collusive tracker payoffs exhibit an upward trend while ii. consumer surplus experiences a concomitant decline with stronger tracker presence. Instead, in the best-case scenario for consumers where collusion is prevented, consumers are worst of in a mixed market, where they often face a single tracker who always matches their price.

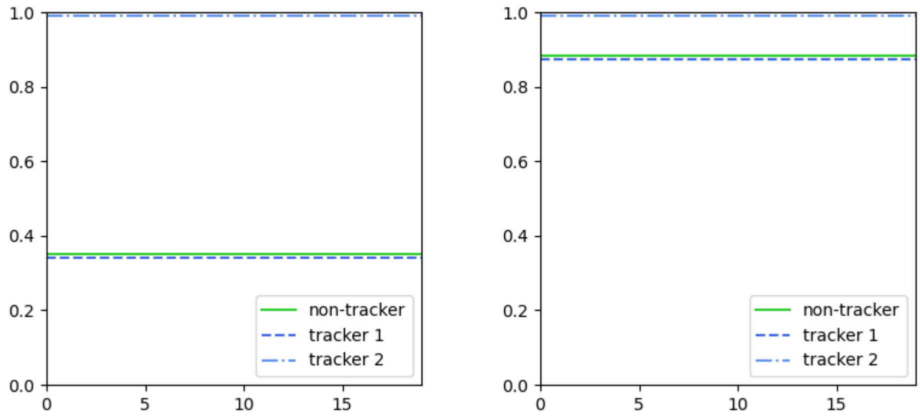


Figure 4: Example price paths under asymmetric collusion.

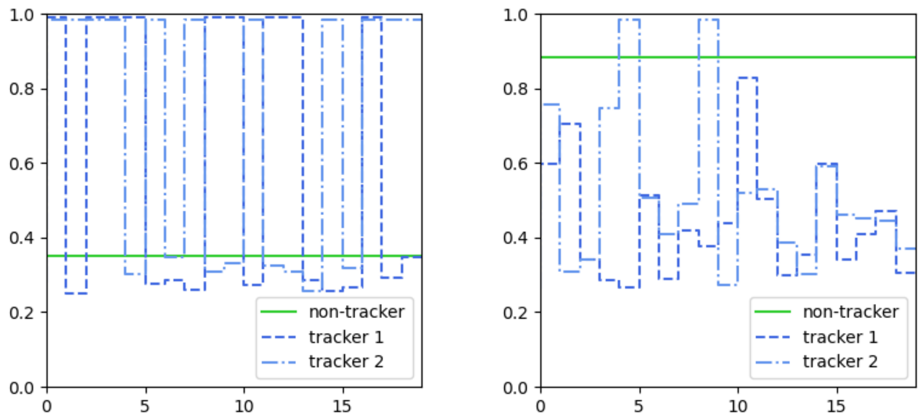


Figure 5: Example price paths without tracker collusion.

Figures 4–6 describe example price paths for the initial time periods of $t = 1, 2, \ldots, 20$ under asymmetric collusion, no collusion, and symmetric collusion, respectively. In each figure, the lhs panel exemplifies parameters $(\tau, \mu) = (0.25, 0.50)$ while the rhs panel is an illustration of parameters $(\tau, \mu) = (0.75, 0.50)$.

[19]Introducing firm heterogeneity as in Narasimhan (1988) to our model would be straightforward. The firms with the smallest consumer base would then have the lowest collusion thresholds.

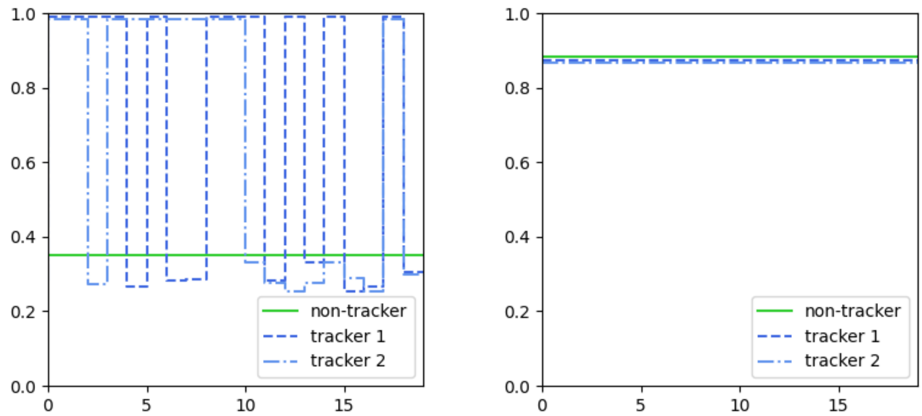


Figure 6: Example price paths under symmetric collusion.

The pattern of pricing looks quite different in each case, possibly allowing to use it as an indicator of collusion.

(a) *Under asymmetric collusion*, lowest prices are tied and the highest price remains at the monopoly price level, resulting in persistent price variation between the prices set by the high-payoff and low-payoff tracker.

(b) *Without tracker collusion*, prices appear scattered. A tracker is more likely to set the monopoly price when a non-tracker offers a lower price. A tracker may also choose a random price from below the best price set by a non-tracker.

(c) *Under symmetric collusion*, prices are tied when a non-tracker offers a high price ($p^0 > c$) but, when the price of a non-tracker is low enough ($p^0 \leq c$), a tracker either offers the monopoly price or a lower mixed price.

5 Regulatory implications

Price monitoring and price algorithms are currently under scrutiny for their potentially significant role in facilitating collusion in electronic marketplaces. We contribute to this discussion by offering a theory of pricing with heterogeneous price technologies and showing how differences among firms impinge on pricing. In general, we find that the presence of trackers in a market is harmful to consumers, regardless of whether the trackers are colluding. On a more positive note, we observe that adverse effects of tracking are mitigated by non-trackers, whose prices bound the prices set by trackers. Non-trackers can also make tracker collusion harder to sustain by limit-price strategies.

The existing literature has suggested several solutions to algorithmic price monitoring and collusion, such as modifying the algorithm learning process (Asker et al., 2024) and implementing a price history-based prominence order (Johnson et al., 2023). However, there is less work on indicators of tracker market impact that could guide such intervention. Our research suggests that price dispersion could function as an empirical test for assessing the influence of tracking on market prices.[20]

Comparison of triopoly price patterns in Figures 4–6 shows that prices are clearly less dispersed with more trackers. Without tracker collusion, lowest prices are distinct. With few trackers, the distribution of prices is almost bimodal as trackers either offer prices below the lowest non-tracker price or at unity. With many trackers, this dichotomous feature disappears as non-trackers then offer higher prices, allowing trackers to reduce their prices. Tightly clustered prices can be a sign of collusion. In cases of asymmetric collusion, the lowest prices are closely matched, while in symmetric collusion, it is the highest prices that are tied. The range of market prices is decreasing in τ.

While firms only apply mixed strategies to avoid being undercut by rivals, price dispersion also helps consumers by decreasing their price payments. Our findings indicate that firms adopt dispersed price strategies when uncertain about their ability to respond to market prices. This finding suggests that reducing pricing frequency may be used to propagate such uncertainty with the goal to amplify price dispersion and increase consumer surplus – impact could be first tested in the lab.[21]

6 Conclusion

We conclude by discussing some extensions of our model:

Non-tracker information. A reasonable conjecture is that without market entry non-trackers eventually discern whether they are facing one tracker or they face asymmetric collusion. Specifically, if non-trackers determine with certainty that a tracker will match any price they offer, non-trackers forego competing over shoppers and set the monopoly price to target captives.

However, if non-trackers learn that rival trackers have difficulties in implementing optimal asymmetric collusion, they will only price beneath the collusion threshold, which allows them to escape direct tracker competition. Further, if there are multiple non-trackers in the market, the optimal non-tracker price strategy is mixed – equilibrium price dispersion hence survives.

Tracker uncertainty. When a tracker is uncertain whether it is competing against a non-tracker or another tracker, different price strategies may arise, with or without a

[20]Berck et al. (2008) showcases such an empirical test. However, some of our proposed price patterns may not be discerned by standard variance metrics and may require closer analysis to separate price variance between cutoffs and beneath a cutoff.

[21]Efficient regulation would trade off consumer surplus with the ability to respond to market shocks (Miklós-Thal and Tucker, 2019; Wang and Werning, 2022), and the impact on search and advertising (Robert and Stahl, 1993; Ater and Rigbi, 2023).

period of learning. Duopoly collusion suffices to illustrate some potential price patterns:

If a tracker is reasonably certain that it is competing against a non-tracker (τ is low), it could initially match the competing price. If the rival turns out to be another tracker, it could reciprocate by undercutting back. At this point both firms recognize that they are engaged in a tracker-tracker competition. Tracker optimal play thus involves initiating the collusive equilibrium.

Instead, if a tracker believes it is likely competing with just trackers (τ is high), it could begin by setting the monopoly price. This strategy allows non-trackers to pool with trackers at collusive monopoly prices. Allowing such pooling is optimal when either the expected benefit of revealing a non-tracker is low or the expected cost of retaliation by other trackers is substantial.

When to make purchases? We assume that consumers make their purchases at the point of entry. This assumption in innocuous if trackers are colluding, as prices remain constant in equilibrium, providing consumers no incentive to delay their purchases. However, when trackers' price strategies are randomized, patient consumers optimally wait for low enough prices, rather than buy when the market prices are high.

Who becomes a tracker? An interesting partially unanswered question is which firms become trackers. Tracking could correlate with other things, such as a firm's production cost, possibly confusing our welfare implications. The literature only shows that firms differ in their menu costs (Anderson et al., 2015; Gorodnichenko and Weber, 2016), but what drives these costs, and how are they evolving? There is no consensus in the associated literature on sticky prices. Research shows that large multiproduct firms (Bhattarai and Schoenle, 2014), better informed firms (Yang, 2022), and firms in certain industry sectors (Gautier and Le Bihan, 2022) have lower menu costs. The role of input-output linkages is studied in Pasten et al. (2020) and rational inattention elaborated in Yang (2022).

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Appendix

Proof of Remark 1

As non-trackers set their price at $t = 0$, pricing game becomes static if $\tau = 0$. In a duopoly, the symmetric equilibrium is derived by Varian (1980) and asymmetric equilibria ruled out by Baye et al. (1992). For completeness we briefly describe the main properties of equilibrium. Steps 1-6 assume that $\mu \in (0, 1)$ while Step 7 studies case $\mu = 0, 1$.

Step 1. Mixed strategies.

Suppose firms use pure pricing strategies p^i and p^j.

If $p^i < p^j$, firm i's payoff is $\frac{1+\mu}{2}p^i$ while firm j's payoff is $\frac{1-\mu}{2}p^j$. A profitable deviation exists. Firm i can increase its payoff by setting a price $p' \in (p^i, p^j)$ between p^i and p^j. This does not affect its demand.

If $p^i = p^j$, firms are tied and each firm's payoff is $\frac{1}{2}p^i = \frac{1}{2}p^j$. A profitable deviation exists. Firm i can increase its payoff by offering a slightly lower price $p' \in (p^i - \epsilon, p^i)$. This raises its demand from $\frac{1}{2}$ to $\frac{1+\mu}{2}$.

By letting F^i denote the distribution function of prices set by firm i, we can define the payoff of the competing firm j in a mixed equilibrium as

$$\pi^j(p^j, F^i) = \left(\frac{1-\mu}{2} + \mu(1 - F^i(p^j)) + \frac{\mu}{2}\mathbf{1}(p^j = p^i)\right) p^j.$$

Step 2. Same support.

Suppose that $\text{supp}F^i = \text{cl}\{p|f^i(p) > 0\}$ and $\text{supp}F^j = \text{cl}\{p|f^j(p) > 0\}$ do not overlap. Consider a price p^i such that $p^i \in \text{supp}F^i$ and $p^i \notin \text{supp}F^j$. Because p^i belongs to $\text{supp}F^j$ but does not belong to $\text{supp}F^j$, there exists an interval of higher prices $[p^i, p^i + \epsilon]$ that neither belong to $\text{supp}F^j$. A profitable deviation thus arises from p^i to $p^i + \epsilon$ because the demand of firm i is constant for all $p \in [p^i, p^i + \epsilon]$ as $(1 - F^i(p)) = (1 - F^i(p^i))$ and $\mathbf{1}(p = p^j) = 0$. We thus conclude that $\text{supp}F^i = \text{supp}F^j$.

Step 3. Same payoffs.

Suppose that $\pi^i < \pi^j$. Let $\underline{p} = \inf \text{supp}F$. Notice that $\pi^j = \lim_{p \to \underline{p}+} \pi^j(p, F^i) = \lim_{p \to \underline{p}-} \pi^i(p, F^j)$ as, otherwise, a profitable deviation from above to below $\underline{p}$ to increase demand from shoppers arises. But this implies that $\pi^i < \pi^j = \lim_{p \to \underline{p}+} \pi^j(p, F^i) \leq \lim_{p \to \underline{p}-} \pi^i(p, F^j)$, which gives a contradiction.

Step 4. No gaps.

Suppose that F^i has a gap $(p^i, p^i + g)$. But then firm j can increase its payoff by deviating from p^i to $p^i + g$ as its probability of selling to shoppers stays constant for $p \in (p^i, p^i + g)$.

Step 5. No atoms.

Suppose that both firms have an atom at $p^i \in \text{supp}F$. Then a profitable deviation arises because each firm can increase its profit by reducing its price from p^i to $p^i - \epsilon$, thus increasing its demand from shoppers from $\frac{\mu}{2}$ to μ.

Suppose that only firm i has an atom at $p^i \in \text{supp}F$. Let $\overline{p} = \sup \text{supp}F$.

If $p^i < \overline{p}$, firm j has a profitable deviation from $p^i + \epsilon$ to $p^i - \epsilon$, which increases its demand from shoppers.

If $p^i = \overline{p}$, firm i's payoff is $\pi = \frac{1-\mu}{2}\overline{p}$ while firm j's payoffs is $\pi = \left(\frac{1-\mu}{2} + \mu\alpha\right)\overline{p}$, which gives a contradiction.

Step 6. Highest and lowest prices.

As there are no atoms, a firm's payoff is defined by $\pi = \frac{1-\mu}{2}\overline{p} = \left(\frac{1-\mu}{2} + \mu\right)\underline{p}$.

Since a firm only attracts captives at the highest price, it profits form setting an even higher price unless $\overline{p} = 1$.

Because setting $\overline{p} = 1$ and $\underline{p}$ must give a firm the same payoff $\pi = \frac{1-\mu}{2}1 = \left(\frac{1-\mu}{2} + \mu\right)\underline{p}$, the lowest price is $\underline{p} = \frac{1-\mu}{1+\mu}$.

Step 7. Boundary parameter values

If $\mu = 0$, Remark 1 holds trivially in the sense that all prices are monopoly prices. If $\mu = 1$, Remark 1 is satisfied trivially since firms price à la Bertrand. However, prices are not randomized over $[0, 1]$ as claimed by Remark 1. $\square$

Proof of Remark 2

Step 1. Stage game equilibrium

The stage game setup is the same as in the game between non-trackers described in Varian (1980). Remark 1 shows that the stage game has a unique mixed equilibrium where a firm's equilibrium payoff is $\frac{1-\mu}{2}$.

This payoff is also a firm's minmax payoff as firm i's best response is to undercut any $p^j > \underline{p}$ and set the monopoly price if $p^j \leq \underline{p}$. Thus, a tracker can always secure the payoff of $\frac{1-\mu}{2}$ by setting the monopoly price unity.

Step 2. Folk theorem

For a large enough discount factor δ, any average expected per-period payoffs $\pi^i + \pi^j \leq 1$ such that $\pi^i, \pi^j \in (\frac{1-\mu}{2}, 1 - \frac{1-\mu}{2})$ can be sustained by the threat or reverting to the equilibrium of the stage game given by Varian (1980). Under these assumptions, equilibrium play yields a payoff of $\frac{\pi^i}{1-\delta}$ whereas a deviation gives at most $\frac{1+\mu}{2} + \frac{\delta}{1-\delta}\frac{1-\mu}{2}$ (a payoff of $\frac{1+\mu}{2}$ in the current period and a payoff of $\frac{1-\mu}{2}$ in any future period). Thus, the deviation is not profitable if

$$\begin{aligned}
\frac{\pi^i}{1-\delta} &\geq \frac{1+\mu}{2} + \frac{\delta}{1-\delta}\frac{1-\mu}{2} \\
\pi^i &\geq (1-\delta)\frac{1+\mu}{2} + \delta\frac{1-\mu}{2} \\
\delta &\geq \frac{\frac{1+\mu}{2} - \pi^i}{\frac{1+\mu}{2} - \frac{1-\mu}{2}}. \qquad (12)
\end{aligned}$$

For symmetric collusive equilibria, the firms split the market equally, so that any price $p^i = p^j$ that generates a payoff π^i strictly above $\frac{1-\mu}{2}$ is incentive compatible for δ satisfying (12).

In asymmetric equilibria, $p^i < p^j$ in certain periods. However, the average expected per-period payoff π^i of a firm must always strictly exceed $\frac{1-\mu}{2}$. Namely, if $\pi^j \leq \frac{1-\mu}{2}$ for $p^j > p^i > \underline{p}$, firm j has a profitable deviation to a price just below p^i for any $\delta < 1$. Asymmetric collusion therefore requires that, if $p^i < p^j$ in the current period, then $p^i > p^j$ in some future period, to average out the payoffs of firms.

If $p^j = 1(> p^i)$, firm j's current payoff is $\frac{1-\mu}{2}$. By varying p^i from $\underline{p}$ to (just below) 1, firm i's current payoff can be varied from $\frac{1-\mu}{2}$ to (just below) $1 - \frac{1-\mu}{2}$ for any δ satisfying (12).

Step 3. Boundary parameter values

If $\mu = 0$, Remark 2 holds trivially in the sense that all prices are monopoly prices. However, there does not exist a continuum of equilibria as claimed by Remark 2. If $\mu = 1$, Remark 2 is satisfied since the firms in the stage game price à la Bertrand. $\square$

Proof of Proposition 1

In a duopoly, two trackers play a repeated game where the stage game equilibrium is as in Varian (1980). The proof is extension of those of Remark 1 for the stage game and Remark 2 for the repeated game.

Step 1. Stage game

Consider the equilibrium in a stage game where trackers commit to prices simultaneously. Denote the distribution function of equilibrium prices that a tracker j offers by G^j. The payoff of tracker i is thus

$$\pi^i(p^i) = \frac{1-\mu}{2}p^i + \mu p^i(1 - G^j(p^i)) + \frac{\mu}{2}p^i\mathbf{1}(p^i = p^j),$$

which gives

$$1 - G^j(p^i) = \frac{\pi^i - \frac{1-\mu}{2}p^i - \frac{\mu}{2}p^i\mathbf{1}(p^i = p^j)}{\mu p^i}.$$

Reversing the roles of firms, we can see that the distribution function of equilibrium prices set by tracker i is given by

$$1 - G^i(p^j) = \frac{\pi^j - \frac{1-\mu}{2}p^j - \frac{\mu}{2}p^j\mathbf{1}(p^j = p^i)}{\mu p^i}.$$

Because firm payoffs are symmetric, $G^i = G^j$ unless $\pi^i \neq \pi^j$ or the distribution of mass points differs between firms.

But neither is possible by Remark 1 Steps 3 and 5, showing $G^i = G^j = G$. This equilibrium price distribution is

$$1 - G(p^i) = \frac{\pi - \frac{1-\mu}{2}p^i}{\mu p^i}.$$

Step 2. Repeated game

If each firm expects that its rival plays the independently repeated equilibrium of the stage game in each period, neither firm has an incentive to deviate from this strategy. As shown in the proof of Remark 1 Step 6, a firm's unique stage game payoff is $\frac{1-\mu}{2}$. The present discounted value of tracker payoffs is given by the geometric sum

$$\frac{1-\mu}{2}(1 + \delta + \delta^2 + \dots) = \frac{1-\mu}{2(1-\delta)}.$$

Because a firm can always guarantee itself at least the per-period payoff of $\frac{1-\mu}{2}$ by selling to its captive non-shoppers at the monopoly price, there exists no equilibrium where the present discounted firm payoffs would remain lower than $\frac{1-\mu}{2(1-\delta)}$. Because consumer surplus is $CS = 1 - \pi^i - \pi^j$, the described equilibrium is consumer-optimal. $\square$

Proof of Proposition 2

The proof extends the proof of Remark 2.

Because the present discounted value of total market demand equals $\frac{1}{1-\delta}$, the sum of firm payoffs $\Pi^i + \Pi^j$ cannot be larger than that, which firms can attain by setting $p^i = p^j = 1$ in each time period $t = 1, 2, \ldots$

This collusive monopoly equilibrium gives both firms the payoff of $\frac{1}{1-\delta}\left(\frac{1-\mu}{2} + \frac{\mu}{2}\right) = \frac{1}{2(1-\delta)}$ as each firm sells to its captives and one half of the shoppers. Monopoly payoffs are thus divided equally between firms.

Collusion can be sustained by the threat of reverting to the repeated Varian (1980) equilibrium upon a deviation of a firm. A deviation to undercut the rival gives at best

$$1 - \frac{1-\mu}{2} + \delta \frac{1-\mu}{2(1-\delta)},$$

while collusion at the monopoly price yields the payoff of

$$\frac{1}{2(1-\delta)}.$$

Two trackers are thus able to collude for all $\delta \geq \frac{\frac{1+\mu}{2} - \frac{1}{2}}{\frac{1+\mu}{2} - \frac{1-\mu}{2}} = \frac{1}{2}$. $\square$

Proof of Proposition 4

Step 1. Atom.

Optimal prices satisfy $p \in (\underline{p}, 1)$, where $\underline{p}$ is defined by $\frac{1-\mu}{3} = (\frac{1-\mu}{3} + \mu)\underline{p}$. By pricing at $p \in (p^0, 1]$, a tracker obtains $\frac{1-\mu}{3}p$, and by pricing at $p \in [\underline{p}, p^0]$, its payoff equals $(\frac{1-\mu}{3} + \mu(1 - L(p)))p$. As pricing at $p \in (p^0, 1)$ is dominated by pricing at $p = 1$, a tracker never chooses $p \in (p^0, 1)$.

Thus, the upper bound of L is either at p^0 or 1. There cannot exist a mass point at p^0 in a symmetric tracker equilibrium L because a tracker then benefits from deviating from p^0 to $p^0 - \epsilon$ to augment its demand from shoppers. The existence of a mass point at unity cannot be overruled.

If the upper bound of L is at 1, tracker payoff is $\frac{1-\mu}{3}$. If the upper bound of L is at p^0, this payoff equals $(\frac{1-\mu}{3})p^0$. Thus, we can see that the upper bound must be one and there must exist a mass point $\alpha(p^0) = 1 - L(p^0)$ at unity to equalize the payoffs at 1 and p^0. This mass point is given by

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu(1 - L(p^0))\right) p^0 \tag{13}$$

Step 2. Discounted prices.

As the payoffs of a tracker are the same as in Varian (1980) for $p < p^0$, standard logic outlined in the proof of Remark 1 suggests that discounted prices $p < p^0$ are randomized over an interval support of $[\underline{p}^1, p^0]$, according to continuous L, which is given by

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu(1 - L(p))\right) p.$$

The support of L is thus $[\underline{p}^1, p^0] \cup \{1\}$. The probability that a tracker matches a price p^0 is given by $1 - (1 - L(p^0))^2$, where $L(p^0)$ solves (13).

As shown by (9) and (10) $\underline{p}^1 < \underline{p}^0$ while $\alpha(p) \to 1$ as $p \to \underline{p}^1$. Hence, a tracker never puts all mass on the price one on the equilibrium path.

Step 3. Lowest prices.

According to Proposition 4, $\alpha(p) = \frac{1-\mu}{3\mu}\frac{1-p^0}{p^0}$. By inserting this into (10) defining $\underline{p}^0$, we can see that

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu\left((1-\tau)^2 + \tau^2\left(\frac{1-\mu}{3\mu}\right)^2\left(\frac{1-\underline{p}^0}{\underline{p}^0}\right)^2\right)\right)\underline{p}^0,$$

which is a second-order-equation in $\underline{p}^0$

$$\left(1 + \frac{3\mu}{1-\mu}(1-\tau)^2 + \tau^2\frac{1-\mu}{3\mu}\right)(\underline{p}^0)^2 - \left(1 + 2\tau^2\frac{1-\mu}{3\mu}\right)\underline{p}^0 + \tau^2\frac{1-\mu}{3\mu} = 0$$

with roots given by

$$\underline{p}^0 = \frac{1 + 2\tau^2\frac{1-\mu}{3\mu} \pm \sqrt{\left(1 + 2\tau^2\frac{1-\mu}{3\mu}\right)^2 - 4(1 + \frac{3\mu}{1-\mu}(1-\tau)^2 + \tau^2\frac{1-\mu}{3\mu})\tau^2\frac{1-\mu}{3\mu}}}{2(1 + \frac{3\mu}{1-\mu}(1-\tau)^2 + \tau^2\frac{1-\mu}{3\mu})}. \tag{14}$$

The discriminant is

$$1 - 4(1-\tau)^2\tau^2 > 0, \text{ for } \tau \in (0,1).$$

The equation (14) defining $\underline{p}^0$ has two (or one) positive roots in $(0,1)$ of which we select the higher one. (If we chose the lower one, then the higher price would not be supported in the equilibrium, as its use is only optimal when all shoppers buy from the firm.) $\square$

Proof of Proposition 5

A non-tracker either faces two trackers, one tracker and one non-tracker, or two non-trackers. Proposition 4 demonstrates that the probability that neither of the two trackers matches a non-tracker's price p is $\alpha(p)^2 = (1 - L(p))^2$. Because attracting shoppers is profitable at all prices that exceed $\underline{p}$, the lowest price a non-tracker is willing to offer, one tracker always matches the lowest of two non-tracker prices. Denoting the symmetric equilibrium price distribution of non-trackers by K, the probability that neither of two competing non-trackers undercuts a price p is $(1 - K(p))^2$.

Merging these results, the payoff of a non-tracker who offers price p can be expressed as

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu\left((1-\tau)^2(1-K(p))^2 + \tau^2\alpha(p)^2\right)\right)p, \tag{15}$$

where $\alpha(p) \to 1$ as $p \to \underline{p}$ and $\alpha(p) \to 0$ as $p \to 1$. Rearranging (15) allows solving for K as

$$\begin{aligned}
K(p) &= 1 - \sqrt{\frac{1-\mu}{3\mu(1-\tau)^2}\frac{1-p}{p} - \frac{\tau^2}{(1-\tau)^2}\alpha(p)^2}, \\
&= 1 - \frac{1}{1-\tau}\sqrt{\frac{1-\mu}{3\mu}\frac{1-p}{p} - \tau^2\left(\frac{1-\mu}{3\mu}\frac{1-p}{p}\right)^2} \\
&= 1 - \frac{1}{1-\tau}\sqrt{\alpha(p) - \tau^2\alpha(p)^2}. \quad \square
\end{aligned}$$

Proof of Proposition 7

1. For $p^0 \leq c$, symmetric tracker collusion cannot be sustained. The equilibrium price distributions of two trackers and non-trackers are hence L and K, given by Propositions 5 and 4, respectively.

2. For $p^0 > c$, non-tracker pricing allows for symmetric tracker collusion at p^0. As a result, any price set by a non-tracker is matched by a tracker, defining the payoff of a non-tracker who offers $p \geq c$ as
$$\left(\frac{1-\mu}{3} + \mu(1-\tau)^2\left(1 - M(p)\right)^2\right)p.$$

3. The lower bound on non-tracker prices either satisfies $\underline{p}^0 \leq c$ or $\underline{p}^0 > c$. According to equation (11), $c = \frac{1-\mu}{1-\mu+3\mu/2}$, while Proposition 4 postulates that $\alpha(p) = \frac{1-\mu}{3\mu}\frac{1-p}{p}$. Combining these findings, we note that $\alpha(c) = \frac{1-\mu}{3\mu}\frac{3\mu/2}{1-\mu} = \frac{1}{2}$.

i. Suppose $\underline{p}^0 \leq c$, which gives

$$\underline{p}^0 := \frac{1-\mu}{1+\mu\left(3(1-\tau)^2 + 3\tau^2\alpha(\underline{p}^0)^2 - 1\right)} \leq c = \frac{1-\mu}{1+\mu/2}.$$

As $\underline{p}^0 \leq c$ implies $\alpha(\underline{p}^0)^2 \geq \alpha(c)^2 = 1/4$, a sufficient condition for $\underline{p}^0 \leq c$ is

$$\frac{1-\mu}{1+\mu\,(3(1-\tau)^2 + 3\tau^2/4 - 1)} \leq \frac{1-\mu}{1+\mu/2}, \Leftrightarrow \tau \leq \frac{4-\sqrt{6}}{5} \approx 0.31.$$

ii. Suppose $\underline{\underline{p}}^0 > c$, which gives

$$\underline{\underline{p}}^0 := \frac{1-\mu}{1+\mu\,(3(1-\tau)^2 - 1)} > \frac{1-\mu}{1+\mu/2}, \text{ iff } \tau > 1 - \frac{1}{\sqrt{2}} \approx 0.29,$$

allowing to sustain multiple equilibria for parameters $1 - \frac{1}{\sqrt{2}} < \tau \leq \frac{4-\sqrt{6}}{5}$.[22]

4. For $\underline{p}^0 < c$, our results indicate further that there must be a gap $(c, c+g)$ in the equilibrium distribution of prices set by non-trackers:

The gap at $(c, c+g)$ arises because pricing at $c - \epsilon$, where $\epsilon \to 0$, gives a non-tracker a higher payoff of

$$\left(\frac{1-\mu}{3} + \mu\tau^2\alpha(c)^2 + \mu(1-\tau)^2\,(1 - M(c))^2\right) c,$$

whereas setting a slightly higher price $c + \epsilon$, where $\epsilon \to 0$, provides a non-tracker the smaller payoff of

$$\left(\frac{1-\mu}{3} + \mu(1-\tau)^2\,(1 - M(c))^2\right) c.$$

Therefore, a profitable deviation from right above c to below it exists for any continuous non-tracker pricing strategy M with $(1 - M(c)) > 0$.

Such a deviation obviously fails to arise if c happens to be the highest price offered by a non-tracker. In this case, pricing at $c - \epsilon$, where $\epsilon \to 0$, provides a non-tracker with the payoff of

$$\left(\frac{1-\mu}{3} + \frac{\mu\tau^2}{4}\right) c,$$

because setting higher prices implies that a non-tracker loses shoppers with two non-trackers. As the monopoly price provides the payoff of $\frac{1-\mu}{3}$, a tracker prefers it over pricing right below c if

$$\frac{1-\mu}{3} > \left(\frac{1-\mu}{3} + \frac{\mu\tau^2}{4}\right)\frac{1-\mu}{1+\mu/2},$$

[22] As collusion provides a higher tracker payoff, the tracker optimal equilibrium has $\underline{\underline{p}}^0 > c$ (not $\underline{p}^0 \leq c$).

which holds as a tautology. Some might still think it possible that non-trackers may wish to commit a lower *highest* price $\overline{p} < c$, yielding

$$\left(\frac{1-\mu}{3} + \mu\tau^2\alpha(\overline{p})^2\right)\overline{p} < \left(\frac{1-\mu}{3} + \mu\alpha(\overline{p})\right)\overline{p} = \frac{1-\mu}{3}$$

But this shows that a non-tracker would benefit from deviating from $\overline{p}$ to the monopoly price unity, demonstrating that the sup of non-tracker prices must equal unity also for $\tau < \frac{4-\sqrt{6}}{5}$ and $\underline{p}^0 < c$.

A price gap means that $M(c) = M(c+g)$. The size of the gap is given by the payoff equality condition between $p = c$ and $p = c + g$:

$$\left(\frac{1-\mu}{3} + \mu\tau^2\alpha(c)^2 + \mu(1-\tau)^2\left(1-M(c)\right)^2\right)c = \left(\frac{1-\mu}{3} + \mu(1-\tau)^2\left(1-M(c+g)\right)^2\right)(c+g)$$

$$\left(\frac{1-\mu}{3} + \mu\tau^2\alpha(c)^2 + \mu(\alpha(c) - \tau^2\alpha(c)^2)\right)c = \left(\frac{1-\mu}{3} + \mu(\alpha(c) - \tau^2\alpha(c)^2)\right)(c+g)$$

$$\left(\frac{1-\mu}{3} + \mu\alpha(c)\right)c = \left(\frac{1-\mu}{3} + \mu\frac{1}{2}\right)c = \left(\frac{1-\mu}{3} + \mu(\frac{1}{2} - \tau^2\frac{1}{4})\right)(c+g).$$

This delivers

$$g = \frac{\tau^2\frac{\mu}{4}}{\frac{1-\mu}{3} + \frac{\mu}{2} - \tau^2\frac{\mu}{4}}c,$$

showing that

$$c+g = \left(1 + \frac{\tau^2\frac{\mu}{4}}{\frac{1-\mu}{3} + \frac{\mu}{2} - \tau^2\frac{\mu}{4}}\right)\frac{\frac{1-\mu}{3}}{\frac{1-\mu}{3} + \mu/2} = \left(\frac{1 + \frac{3\mu}{1-\mu}\frac{1}{2}}{1 + \frac{3\mu}{1-\mu}\frac{1}{2}(1-\tau^2\frac{1}{2})}\right)\left(\frac{1}{1 + \frac{3\mu}{1-\mu}\frac{1}{2}}\right) < 1$$

for all $(\mu, \tau) \in (0,1)^2$. Hence, the gap never reaches the monopoly price. □

Proof of Remark 5

$$\left(\mu + \frac{1-\mu}{3}\right)p^0 + \delta\frac{1}{1-\delta}\frac{1-\mu}{3} \leq \frac{1}{1-\delta}\left(\frac{\mu}{3} + \frac{1-\mu}{3}\right)p^0.$$

$$(1-\delta)\left(\mu + \frac{1-\mu}{3}\right)p^0 + \delta\frac{1-\mu}{3} \leq \left(\frac{\mu}{3} + \frac{1-\mu}{3}\right)p^0.$$

$$\left(\mu + \frac{1-\mu}{3}\right)p^0 - \left(\frac{\mu}{3} + \frac{1-\mu}{3}\right)p^0 \leq \delta\left(\left(\mu + \frac{1-\mu}{3}\right)p^0 - \frac{1-\mu}{3}\right).$$

$$\delta \geq \frac{\frac{\mu}{3}p^0}{\left(\left(\mu + \frac{1-\mu}{3}\right)p^0 - \frac{1-\mu}{3}\right)}.$$

Above, $\frac{\frac{\mu}{3}p^0}{\left(\left(\mu+\frac{1-\mu}{3}\right)p^0 - \frac{1-\mu}{3}\right)} < 1$ if $p^0 > c = \frac{1-\mu}{1-\frac{1}{2}\mu}$, which shows that the condition is satisfied for all high enough values of δ. □