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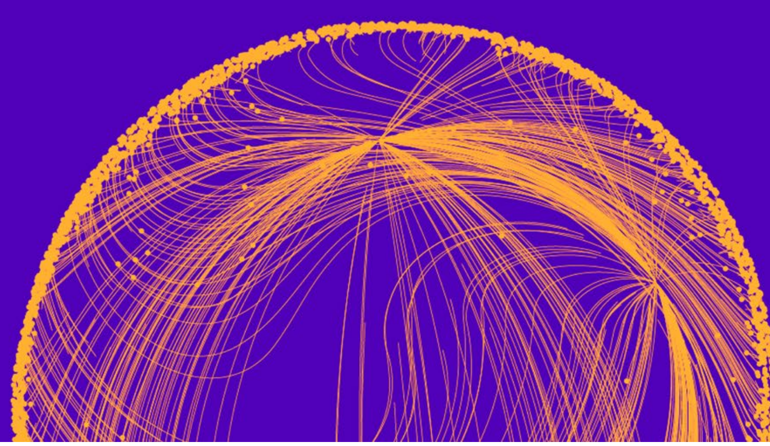

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Helsinki GSE Discussion Papers 57 · 2026

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ISBN 978-952-7543-56-6 (PDF)
ISSN 2954-1492

Helsinki GSE Discussion Papers:
https://www.helsinkigse.fi/discussion-papers

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PO BOX 21210
FI-00076 AALTO
FINLAND

Helsinki, March 2026

Endogenous Education Duration, Fertility Choice, and Mortality: A Framework for Optimal Population Policy

Tapio Palokangas*

March 9, 2026

Abstract

This paper characterizes optimal education and population policies within a continuous-time overlapping generations (OLG) framework where fertility, schooling duration, and health expenditures are endogenous. In this model, mortality risk is driven by a negative congestion effect from population growth and a positive spillover effect from education duration.

We demonstrate that when labor productivity is tied to schooling length, traditional fiscal instruments – such as taxes on newborns – are insufficient to achieve social efficiency. Instead, we propose a multi-faceted policy framework integrating public health provision, education grants, and a corrective poll tax, supported by government-backed student loans to mitigate liquidity constraints. Our analysis yields three optimality conditions. First, public health care must be expanded until its marginal product equals that of private health care. Second, the student grant must be proportional to the marginal rate of substitution between education and health care, but inversely proportional to the interest rate and cohort size. Third, the poll tax must cover public health costs while reflecting the marginal rate of transformation between congestion and private health care. These findings advocate for a strategic policy pivot: shifting from the Pigouvian management of population quantity toward fostering the quality and longevity of human capital.

Journal of Economic Literature: J13, I12, I22, H51, H52, D62

Keywords: overlapping generations; endogenous fertility; endogenous mortality; congestion externalities; human capital; public health expenditure; education subsidies; student loans.

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1 Introduction

The nexus between education and population growth has traditionally been examined through two primary lenses. The first conceptualizes education as a continuous process of human capital accumulation, often treated as an investment analogous to physical capital (e.g., Agénor, 2008). The second employs discrete-time overlapping generations (OLG) models in which generation length – and by extension, schooling duration – is assumed to be constant (e.g., Agénor, 2019). Crucially, both frameworks typically overlook the endogeneity of schooling time. This study addresses this gap by developing a continuous-time framework where households determine the duration of their children's education.

Within our framework, a social planner intervenes through taxation, public health care and guaranteed student loans; consequently, government-mandated compulsory education emerges not as an exogenous parameter, but as a special case contingent upon specific policy configurations. By endogenizing the shift from child quantity to quality, this approach builds upon the foundational insights of Unified Growth Theory (Galor and Weil 2000, Galor 2011), which identifies the rising return to human capital as the historical catalyst for the demographic transition.

The existing literature on optimal taxation, most notably Harford (1997, 1998), primarily employs discrete-time models in which altruistic parents determine fertility. In Harford's framework, population growth generates environmental externalities that necessitate a Pigouvian pollution tax coupled with a parental tax on fertility. However, in our continuous-time model, individuals are classified as "students" until their education is complete, creating a distinct phase of dependency. We demonstrate that within this structure, a direct parental tax is both practically difficult to implement and theoretically insufficient to achieve the social optimum. Instead, we show that social efficiency can be restored through a study grant. To resolve the liquidity constraints inherent in a family's initial wealth, this policy must be supported by government-backed student loans.

Similarly, while Jöst and Quaas (2010) advocate for a dual-tax regime targeting both carbon emissions and family size, our framework departs from their approach by endogenizing the duration of education. In this setting, population growth diminishes welfare through congestion – manifesting either as direct physical crowding or as environmental degradation. We demonstrate that while a fertility tax might be sufficient if congestion were the singular externality, the endogeneity of schooling requires a more sophisticated policy mix to calibrate the trade-off between population quality and quantity. This aligns with Galor's (2012) thesis that the demographic transition is not merely a passive byproduct of rising income, but a strategic reallocation of resources toward human capital. Consequently, an optimal policy framework must internalize this structural substitution between family size and educational investment.

A central innovation of this study lies in the modeling of mortality. Traditional approaches often treat mortality as a function of state variables like health capital (Grossman, 1972) or the accumulation of health deficits (Dalgaard and Strulik, 2014). Departing from these stock-based perspectives, we propose that mortality is governed by an error-correction process driven by population congestion and tempered by education. This aligns with extensive empirical research supporting a robust positive correlation between mass education and population health (Caldwell, 1979; Baker et al., 2011). Social epidemiology further suggests that education reduces disease prevalence independently of socioeconomic status (Mackenbach et al., 2008). Guided by this evidence, our model assumes that increased education length reduces mortality over time through a corrective dynamic. This creates a feedback loop similar to that described by Galor and Moav (2002, 2005), where lower mortality extends the "payout period" of human capital, thereby justifying longer schooling durations. In our model, this creates a "health effect" that policymakers must internalize to reach the social optimum.

Finally, while recent work (e.g., He, 2025) analyzes human capital as a tradable asset influenced by status-seeking behavior, such models often overlook the worker-specific nature of human capital and the discrete temporal lags inherent in schooling. In contrast, our framework treats education length as a strategic control variable for both households and the government, capturing the temporal reality of human capital development more accurately than continuous accumulation models.

The remainder of this study is organized as follows: Section 2 outlines the economic structure, focusing on production using capital and educated labor. Section 3 derives household decision-making regarding savings, fertility, and schooling duration. Section 4 examines government behavior and establishes optimal policy rules by comparing the decentralized equilibrium with the social planner's first-order conditions. Section 5 concludes with a summary of findings and policy implications.

2 The economy as a whole

2.1 Population growth

In this model, we treat time t as a continuous variable. Each individual experiences two distinct life stages. During the first stage, which lasts D units of time, the individual pursues education and does not have children. Upon completing their studies, the individual enters the workforce for the remainder of their lifespan, during which they have children, accumulate capital, and invest in the education of their offspring. The population consists of a large number of households, each comprising individuals in both life stages: students and workers. These households share identical preferences and operate under the same educational technology.

Because only workers have children, the *fertility rate* f is defined in proportion to the *working population* N. In fertility-choice models of population growth, it is commonly assumed – primarily for analytical tractability – that all individuals are subject to a uniform mortality rate.[1] In other words, the instantaneous probability of death is assumed to be independent of age. This study adopts the same simplifying assumption.

In the model, the mortality rate of workers, m, remains age-independent but is endogenously affected by congestion, health care and the length of education. For technical reasons, students are assumed to survive until graduation – that is, their mortality rate is zero during studies and m thereafter.[2]

[1] See Razin and Ben-Zion (1975), Becker (1981), Galor and Weil (1996), Stelter (2016), Palokangas (2021), and Lehmijoki and Palokangas (2023).

[2] Francis and Kompas (2001) note that Uzawa's transformation, used in the Appendix to eliminate variable mortality rates from discount factors, is valid only for autonomous systems. If students faced positive mortality, the model would involve non-autonomous differential equations.

An individual born at time $t-D$ contributes to the labor force at time t, and the entire cohort born at time $t-D$, whose size is $f(t-D)N(t-D)$, collectively contributes the quantity $f(t-D)N(t-D)$ to the labor force at time t. Simultaneously, a fraction m of the current workers N dies. Since the net change in the labor force N at time 0 is

$$\dot{N} \doteq \frac{dN}{dt} = f(t-D)N(t-D) - mN, \quad N(0) = N_0, \tag{1}$$

where N_0 is the value of N at time 0.

2.2 Education, population and the supply of labor

We assume that a child's future productivity as a worker, denoted by q, increases with the duration of their education D though at a diminishing rate. This defines the function

$$q(D), \quad q(0) = 0, \quad q' > 0, \quad q'' < 0. \tag{2}$$

With productivity (2), the entire cohort born at time $t-D$, $q(D)f(t-D)N(t-D)$, collectively contributes $q(D)f(t-D)N(t-D)$ to *effective labor* at time t. Since a fraction m of the current workers supplying effective labor dies, the net change in effective labor Z at time t is

$$\dot{Z} = q(D)f(t-D)N(t-D) - mZ, \quad Z(0) = Z_0, \tag{3}$$

where Z_0 is the value of Z at time 0. The student population at time t comprises cohorts born during the interval $\xi \in (t-D, t]$, defined as follows:

$$\int_{t-D}^{t} f(\xi)N(\xi)d\xi. \tag{4}$$

The total population Θ consists of workers N and students (4):

$$\Theta \doteq N + \int_{t-D}^{t} f(\xi)N(\xi)d\xi. \tag{5}$$

Each newborn is assumed to require a fixed amount b of labor for child-rearing at birth. Consequently, a fraction bf of the workforce is allocated to this task, where f denotes the fertility rate. The remaining share $1-bf$ of effective labor Z constitutes the supply of effective labor, L:

$$L = (1-bf)Z. \tag{6}$$

2.3 Production and the accumulation of capital

In the model, there is only one tradable good and capital K is the only tradable asset. The good serves as the *numeraire* whose price is normalized to one. The aggregate output of the good, Y, is produced from capital K and effective labor L through technology

$$Y = F(K, L), \quad F_K \doteq \frac{\partial F}{\partial K} > 0, \quad F_L \doteq \frac{\partial F}{\partial L} > 0,$$
$$F \text{ concave and linearly homogeneous.} \tag{7}$$

To streamline the analysis, the production function (7) is specified net of capital depreciation. Consequently, the interest rate, denoted by r, represents the net rate of return, incorporating the depreciation rate of capital.

Health care consists of two distinct types, differing in delivery and impact:

- *Private health care* is treated as a private good. An input h allocated to a specific worker reduces only that individual's mortality rate m.
- *Public health care* is modeled as a public good. It is scaled to the total working population N, such that the input per worker g reduces the mortality rate across the entire workforce.

Since both inputs are composed of the numeraire good, the economy's total health expenditure is expressed as $(h + g)N$.

Let c denote a worker's consumption and s a student's consumption. Total output (7) is used in the consumption of workers N, cN, the consumption of students (4), $s\int_{t-D}^{t} f(\xi)N(\xi)d\xi$, health expenditures $(h+g)N$, and investment in new capital, $\dot{K} \doteq \frac{dK}{dt}$. Thus, capital K evolves according to

$$\dot{K} = F(K, L) - cN - s\int_{t-D}^{t} f(\xi)N(\xi)d\xi - (h + g)N, \quad K(0) = K_0, \tag{8}$$

where K_0 is the value of capital K at time 0.

2.4 Firms

The representative firm takes the wage w for labor L and the interest rate r paid to capital K as given and maximizes its profit subject to (7),

$$\Pi \doteq Y - L - rK = F(K, L) - wL - rK, \tag{9}$$

by its inputs (K, L). With constant returns to scale, it earns no profit and sets the marginal products of the inputs equal to the input prices:

$$\Pi = 0, \quad F_K = r, \quad F_L = w. \tag{10}$$

2.5 The public sector

Given that individuals are born prior to making education decisions, we assume that the government provides a grant γ to all currently enrolled students (4), excluding newborns fN. We assume that the government finances its health expenditure gN and the student grants $\gamma\left[\int_{t-D}^{t} f(\xi)N(\xi)d\xi - fN\right]$ through a poll tax x levied on the working population N:[3]

$$xN = gN + \gamma\left[\int_{t-D}^{t} f(\xi)N(\xi)d\xi - fN\right]. \tag{11}$$

2.6 Mortality dynamics

Epidemiological research – notably Samet et al. (2000), Pope et al. (2002), and Lehmijoki and Rovenskaya (2010) – establishes that population congestion, denoted by P, exacerbates mortality by intensifying pollution and infection rates. In alignment with these findings, the empirical evidence presented in Section 1 suggests that extended education duration, D, serves as a significant mitigating factor, lowering mortality risk.

Because congestion P, education duration D, private health care per worker h, and public health care per worker g influence mortality with significant temporal lags, the evolution of the mortality rate m must be modeled as an error-correction process:

$$\dot{m} = v\big[M(P, D, h, g) - m\big], \quad \frac{\partial M}{\partial P} > 0, \ \frac{\partial M}{\partial D} < 0, \ \frac{\partial M}{\partial h} < 0, \ \frac{\partial M}{\partial g} < 0,$$
$$v \in (0,1), \ m(0) = m_0, \ M \in (0,1), \ m \in (0,1), \ M \text{ strictly concave}, \tag{12}$$

where v is the constant adjustment parameter, M the driver of mortality, and m_0 the initial mortality rate at time zero.

[3] While the budget could also be balanced via consumption or income taxes, such alternatives would complicate the resulting policy rules without qualitatively altering our findings.

Using the driver function $M(P, D, h, g)$, we define two key concepts that characterize the trade-offs between the factors of mortality:

- The *marginal rate of substitution* ε_D represents the trade-off between education duration D and aggregate private health care $H = hN$:

$$\varepsilon_D \doteq \left|\frac{dH}{dD}\right|_{M \text{ constant}} = N\left|\frac{dh}{dD}\right|_{M \text{ constant}} = N \underbrace{\frac{\partial M}{\partial D}}_{-} \Bigg/ \underbrace{\frac{\partial M}{\partial h}}_{-} > 0. \quad (13)$$

  This indicates the exogenous efficiency gain from human capital; it shows the reduction in required aggregate private health spending H enabled by a marginal increase in education duration D.

- The *marginal rate of transformation* ε_P determines the compensatory spending required due to congestion P:

$$\begin{aligned}\varepsilon_P \doteq \left|\frac{dH}{dP}\right|_{M \text{ constant}} &= N\left|\frac{dh}{dP}\right|_{M \text{ constant}} = N\left|\underbrace{\frac{\partial M}{\partial P}}_{+} \Bigg/ \underbrace{\frac{\partial M}{\partial h}}_{-}\right| \\ &= -N\frac{\partial M}{\partial P} \Bigg/ \frac{\partial M}{\partial h} > 0. \end{aligned} \quad (14)$$

  This reflects the marginal cost of externalities, telling us how much additional aggregate private health care M is necessary to countervail the negative impact of a marginal increase in congestion P.

3 Households

3.1 Utility

Households have compelling incentives to invest in education, recognizing that such investments bolster their children's future earnings while reducing mortality risks. To address potential liquidity constraints in education financing, we assume the government guarantees all student loans. This guarantee ensures that all households can access capital markets at the prevailing interest rate r. Consequently, individual household behaviors align, allowing us to model the economy through a single representative household (hereafter referred to as the household).

In fertility-choice models of population growth, a household derives utility from its member's consumption c and its fertility rate, f.[4] This utility is typically discounted over the foreseeable future by a constant rate of time preference, ρ. In this study, these conventional assumptions are retained, but because a household consists of two distinct types of individuals – workers and students – their consumption must be modeled separately. Therefore, a household's instantaneous utility u is a strictly concave function of a worker's consumption c, a student's consumption s and the fertility rate f as follows:

$$u(c, s, f), \quad u_c \doteq \frac{\partial u}{\partial c} > 0, \quad u_s \doteq \frac{\partial u}{\partial s} > 0, \quad u_f \doteq \frac{\partial u}{\partial f} > 0. \tag{15}$$

Without loss of generality, time zero can be designated as the initial moment for the household. Because the mortality rate m is uniform for all workers independently of the age, the probability of a worker surviving beyond the interval $[0, t)$ is e^{-mt}. Accordingly, by (15), a worker's expected instantaneous utility at time t is $e^{-mt}u(c, s, f)$. Consequently, a worker's expected intertemporal utility at the initial time $t = 0$ is the integral of these instantaneous utilities over for the lifespan $t \in [0, \infty)$, discounted by the constant rate of time preference ρ:

$$U \doteq \int_0^\infty [e^{-mt}u(c, s, f)]e^{-\rho t}dt = \int_0^\infty u(c, s, f)e^{-(\rho+m)t}dt, \tag{16}$$

where $\rho + m$ is the *effective rate of time preference* with mortality.

3.2 Optimal behavior

In the household's budget, saving (= the accumulation of capital K) $\dot{K}$ is equal to the returns paid to capital, rK, plus wages $wL = (1 - bf)wZ$ [cf., (6)] plus student grants $g\left[\int_{t-D}^t f(\xi)N(\xi)d\xi - fN\right]$ [cf., (4)] minus the workers' consumption cN, the students' consumption $s\int_{t-D}^t f(\xi)N(\xi)d\xi$, private health care hN and the poll taxes xN. This is equivalent to

$$\dot{K} = rK + (1 - bf)wZ - (s - \gamma)\int_{t-D}^t f(\xi)N(\xi)d\xi - \gamma fN - (c + x + h)N. \tag{17}$$

To interpret the results, we define three fundamental concepts:

[4]Cf. Razin and Ben-Zion (1975), Becker (1981), Galor and Weil (1996), Stelter (2016), Palokangas (2021), and Lehmijoki and Palokangas (2023).

- The *effective rate of time preference* $r + m$, represents the discount factor for the accumulation of effective labor. It combines the prevailing interest rate r with the mortality rate m, where the latter serves as the depreciation rate of effective labor.

- We define *a worker's human capital*, $\frac{1}{r+m}\frac{wL}{N}$, as the present value of the labor income stream per worker, $\frac{wL}{N}$, discounted by the effective rate of time preference $r + m$.

- We define the *marginal revenue of education duration*, z, as the product of the relative productivity gain, $\frac{q'}{q}$, and human capital, $\frac{1}{r+m}\frac{wL}{N}$, discounted by the interest rate r. Combining this definition with (2), we obtain z as a decreasing function of education duration D:

$$z(D, r, m, w, L) \doteq \frac{1}{r}\frac{1}{r+m}\frac{q'(D)}{q(D)}\frac{wL}{N} \quad \text{with}$$
$$\frac{\partial z}{\partial D} = \frac{1/r}{r+m}\frac{wL}{N}\frac{d}{dD}\left[\frac{q'(D)}{q(D)}\right] = \left[\frac{q''}{q} - \left(\frac{q'}{q}\right)^2\right] < 0. \tag{18}$$

The household takes as exogenously given the wage w, the interest rate r, the student grant γ, the poll tax x, population congestion P and public health spending per worker, g. It maximizes utility (16). It chooses worker consumption c, student consumption s, the fertility rate f and its health spending per worker, h, to maximize its utility. In addition, it chooses the education duration D for each newborn at the moment of birth. These choices are made subject to the dynamics of the working population N (1), effective labor Z (3), the capital stock K (17) and mortality (12). The solution of this problem yields the first-order steady-state conditions (Appendix A)

$$f = m, \quad r = \rho + m, \quad q = Z/N, \quad u_s/u_c = rmD, \quad s + bwq = u_f/u_c, \tag{19}$$
$$z + \gamma = s, \tag{20}$$
$$\left(\frac{r}{v} + 1\right)\frac{r+m}{\frac{\partial M}{\partial h}} = c + x + h + sm - \frac{wL}{N} - \frac{r+m}{r}\frac{u}{u_c}. \tag{21}$$

According to equations (19), several key equilibrium conditions emerge:

Demographic balance: The fertility rate f is equal to the mortality rate m.

Labor productivity: A worker's efficiency q aligns with the effective labor per worker, expressed as Z/N.

Optimal education investment: The marginal rate of substitution (MRS) between a student's and a worker's consumption, u_s/u_c, is equated to the interest costs associated with a cohort's education, rmD.

Optimal fertility: Since a student does not earn labor income, their consumption s represents a net cost to the household. The MRS between fertility and a worker's consumption, u_f/u_c, equals the total cost of an additional child, which comprises the student's consumption s and the direct birth costs bwq.

According (18), an increase in education duration, D, diminishes its marginal revenue z. According to (20), households extend D until the total marginal benefit of D – comprising the marginal return z and the student grant γ – equates to the marginal cost of D, which is represented by the student's consumption s.

For interpretation, equation (21) is rearranged as follows:

$$\frac{1}{r+m}\frac{wL}{N}+\frac{1}{r}\frac{u}{u_c}=\frac{c+x+h+sm}{r+m}-\left(\frac{r}{v}+1\right)\Big/\frac{\partial M}{\partial h},$$

where the terms represent the following:

- $\frac{1}{r+m}\frac{wL}{N}$ is the discounted labor income per worker, where $r+m$ reflects the effective rate of time preference for a mortal individual,
- $\frac{1}{r}\frac{u}{u_c}$ denotes the discounted flow of utility in consumption units,
- $\frac{c+x+h+sm}{r+m}$ represents the discounted total expenditures per worker, and
- the final term $-\left(\frac{r}{v}+1\right)/\frac{\partial M}{\partial h}>0$ captures the marginal cost of mortality. Specifically, it quantifies the additional private health care expenditures $dh>0$ required to offset an increase in the mortality driver, $dM>0$, while maintaining a constant mortality rate m.

4 The social planner

To derive the first-best solution for public policy, we consider the government as a social planner. In this model, the worker population N serves as a proxy for population congestion: $P = N$.[5]

The social planner chooses private and public health care (h, g), worker consumption c, student consumption s, the fertility rate f, and the duration of education D for each newborn. In this model, Walras' law holds: the government budget constraint (11) is redundant, as it is implicitly satisfied by the equilibrium conditions (8), (9), (10), and (17). Furthermore, since the driver function $M(N, D, h, g)$ in (12) is strictly concave, public policy can be implemented in a second-best framework without lump-sum taxes.

The planner's objective is to maximize the household's utility (16) subject to the labor supply constraint (6) and the dynamics governing the working population N (1), effective labor Z (3), capital K (8), and the mortality rate m (12) where $P = N$. As detailed in Appendix B, this maximization problem yields the following steady-state first-order conditions:

$$f = m = M, \quad F_K = \rho + m, \quad q = \frac{Z}{N}, \quad s + F_L bq = \frac{u_f}{u_c}, \quad u_s = F_K m D u_c, \tag{22}$$

$$\frac{\partial M}{\partial g} = \frac{\partial M}{\partial h}, \tag{23}$$

$$s = \frac{1}{F_K}\left(\frac{F_L L}{N}\frac{q'/q}{F_K + m} + \frac{1}{m}\frac{\partial M}{\partial D}\Big/\frac{\partial M}{\partial h}\right), \tag{24}$$

$$\left[\left(\frac{F_K}{v} + 1\right)(F_K + m) + N\frac{\partial M}{\partial P}\right]\frac{1}{\frac{\partial M}{\partial h}} = c + h + g + sm - \frac{F_L L}{N} - \frac{F_K + m}{F_K}\frac{u}{u_c}. \tag{25}$$

The planner's equilibrium conditions (22) are satisfied by construction, as they are, by the firm's equilibrium conditions (10), identical to the household's equilibrium conditions (19). Condition (23) proves the following:

Proposition 1 *Public health care g should be expanded until its marginal efficiency in reducing long-run mortality M equals that of private health care h. Specifically:* $\frac{\partial M}{\partial g} = \frac{\partial M}{\partial h}$.

[5] Alternative measures, such as output Y [cf. (7)] or aggregate population Θ [cf. (5)], are proportional to N in the steady state and highly correlated in empirical data. Therefore, their inclusion would increase complexity without altering the qualitative results.

Substituting the firms equilibrium conditions (10) and the definitions (18) and (13) into conditions (24) and (25), we obtain

$$s = \frac{1}{r}\left(\frac{wL}{N}\frac{q'/q}{r+m} + \frac{1}{m}\frac{\partial M}{\partial D}\Big/\frac{\partial M}{\partial h}\right) = z + \frac{\varepsilon_D}{rmN}, \tag{26}$$

$$\left[\left(\frac{r}{v}+1\right)(r+m) + N\frac{\partial M}{\partial P}\right]\frac{1}{\frac{\partial M}{\partial h}} = c + h + g + sm - \frac{wL}{N} - \frac{r+m}{r}\frac{u}{u_c}. \tag{27}$$

Combining the household's equilibrium condition (20) with the planner's equilibrium condition (26) yields

$$\gamma = s - z = \frac{\varepsilon_D}{rmN} > 0.$$

This result can be rephrased as follows:

Proposition 2 *The optimal student grant $\gamma = \frac{\varepsilon_D}{rmN}$ is determined by three primary factors. First, it is directly proportional to ε_D, the marginal rate of substitution between education duration and aggregate private health care. Second, it is inversely proportional to the interest rate r, reflecting the discounted future value of human capital. Finally, it is inversely proportional to the cohort size mN, over which the grant's benefits are distributed.*

Combining the planner's equilibrium condition (27) with the household's condition (21) and noting the definition (14), we obtain

$$x = g - N\underbrace{\frac{\partial M}{\partial P}}_{+}\Big/\underbrace{\frac{\partial M}{\partial h}}_{-} = g + \varepsilon_P > 0. \tag{28}$$

This result leads to the following formal characterization:

Proposition 3 *The optimal poll tax $x = g + \varepsilon_P$ is equal to public health expenditure per worker, g, plus the marginal rate of transformation between congestion and aggregate private health spending, ε_P.*

The poll tax x serves a dual fiscal purpose. First, it directly finances public health expenditures per worker g. Second, to internalize the negative externalities of population congestion, the tax must also incorporate the marginal rate of transformation ε_P, representing the social cost of population growth relative to health outcomes.

5 Conclusions

This study characterizes an overlapping generations (OLG) economy where education duration, fertility, and mortality are determined endogenously. Within this framework, household decisions drive fertility rates, the duration of schooling required to enhance future productivity, and the private health expenditures necessary to mitigate mortality. Mortality risk is subject to the negative pressure of population congestion and the positive health externalities generated by both education and public health care.

A primary contribution of this analysis is its departure from conventional modeling constraints. We demonstrate that when schooling duration drives labor productivity, traditional instruments – such as parental taxes on newborns – are suboptimal. To achieve a social optimum, the government must adopt a more comprehensive fiscal framework.

To eliminate distortions caused by liquidity constraints, the state must provide government-backed student loans, ensuring that human capital investment is not hindered by a lack of immediate capital. Because children remain in the educational system longer as productivity demands increase, a "one-size-fits-all" tax at birth is impractical. Instead, we propose education grants indexed to the length of schooling.

A poll tax (or a proportional levy on the working population) is required to internalize the negative externalities of population congestion. Public health care serves a dual purpose: it optimizes the aggregate package of private and public health inputs, and it provides a benchmark for valuing mortality to evaluate the external benefits of education.

Our results yield three central policy mandates. First, public health care provision should expand until its marginal product equals the marginal cost of private health care. If private and public inputs are imperfect substitutes, their strategic combination significantly enhances social welfare.

Since the health externality of education are typically under-internalized, the socially optimal duration of schooling exceeds the laissez-faire equilibrium. Consequently, the optimal student grant should be proportional to the marginal rate of substitution between education and aggregate private health care. Furthermore, as a higher interest rate or a larger cohort size increase the discounted costs of schooling, the grant must be adjusted inversely to these rates to maintain investment incentives.

Since congestion increases mortality, the socially optimal health provision exceeds the aggregate levels seen in a *laissez-faire* scenario. The optimal poll tax should not only cover public health costs per worker but also incorporate the marginal rate of transformation between congestion and private health care to internalize the social cost of population growth.

These findings advocate for a strategic shift from the traditional Pigouvian approach of taxing fertility to manage population quantity. Social efficiency is best achieved by targeting the quality and longevity of human capital. Study grants should be viewed not merely as social transfers, but as strategic tools to counteract population congestion and labor force erosion.

While this model provides a robust framework for endogenous demographics, several avenues for future research remain. First, the assumption of a homogeneous labor force can be relaxed to investigate how grants influence income distribution and the skill premium. Second, international labor migration can be introduced to model "brain drain" or "brain gain" effects as a dynamic game between competing governments. Third, public debt can be incorporated into formal budget constraints to evaluate the long-term viability of these policies within the context of an aging population.

Appendix

A The household's optimum (eqs 19-21)

A.1 Transformation into virtual time

Because, according to (12), the mortality rate m is an endogenous variable, it must be eliminated from the discount factor in the utility function (16) using Uzawa's (1968) transformation

$$\theta(t) = [\rho + m(t)]t \text{ with } dt = \frac{d\theta}{\rho + m(\theta)}. \tag{29}$$

Because from (29) it follows that $\theta(\infty) = \infty$ and $\frac{dt}{d\theta} = \frac{1}{\rho + m(\theta)} > 0$, the function $\theta(t)$ can be interpreted as a *virtual time* variable, allowing all variables to be re-expressed in terms of θ.

A.2 The system in virtual time

According to equation (29), the duration of education in virtual time, denoted by θ, is a function of its counterpart in real time, D, as follows:

$$\theta(D) = [\rho + m(D)]D \text{ with } \theta'(D) = \rho + m(D), \tag{30}$$

where $m(D)$ is the value of the variable m at time D. For notational simplicity, the explicit time argument is omitted from time-dependent variables when it addresses current virtual time $\theta(t)$.[6] In this model, the symbol D plays two distinct roles. First, it represents the duration of education – a control variable selected from the set of constant controls at the current time $\theta(t)$. This variable appears exclusively in the functions $q(D)$ and $\theta(D)$. Second, D also serves as a temporal parameter in expressions such as such as $t - D$, D or $t + D$, where it interacts with other control variables (c, s, f) as well as the state and costate variables $(K, N, Z, \lambda_K, \lambda_N, \lambda_Z)$.

By applying equations (6), (12), and (29), one can reformulate the utility function (16) and with the dynamics of the key state variables – namely, the population of workers (1), effective labor (3), and capital stock (17) – from real time t to virtual time θ as follows:

$$U \doteq \int_0^\infty \frac{u(c, s, f)}{\rho + m} e^{-\theta} d\theta, \tag{31}$$

$$\frac{dN}{d\theta} = \frac{1}{\rho + m}\big[f\big(\theta(t - D)\big)N\big(\theta(t - D)\big) - mN\big]_, \; N(\theta(t)) = N_t, \tag{32}$$

$$\frac{dZ}{d\theta} = \frac{1}{\rho + m}\big[q(D)f\big(\theta(t - D)\big)N\big(\theta(t - D)\big) - mZ\big]_, \; Z(\theta(t)) = Z_t, \tag{33}$$

$$\frac{dm}{d\theta} = \frac{v}{\rho + m}\big[M(P, D, h, g) - m\big]_, \quad m(0) = m_0, \tag{34}$$

$$\frac{dK}{d\theta} = \frac{1}{\rho + m}\Bigg[rK + w(1 - bf)Z + (\gamma - s)\int_{\theta(t-D)}^{\theta(t)} f(\xi)N(\xi)d\xi - (c + x + h + \gamma f)N\Bigg]_, \quad K(\theta(t)) = K_t. \tag{35}$$

[6] That is, $N(\theta(t)) = N$, $Z(\theta(t)) = Z$, $K(\theta(t)) = K$, $m(\theta(t)) = m$, $c(\theta(t)) = c$, $s(\theta(t)) = s$, $f(\theta(t)) = f$, $\lambda_N(\theta(t)) = \lambda_N$, $\lambda_Z(\theta(t)) = \lambda_Z$, $\lambda_K(\theta(t)) = \lambda_K$ and $\lambda_m(\theta(t)) = \lambda_m$.

A.3 The Hamiltonian

The household maximizes the utility function (31) by selecting the control variables (h, c, s, f, D) – of which D from the set of constant controls – subject to the dynamic constraints (32)-(35), given the wage w, the return on capital r, the grant γ, the poll tax x and population congestion P. Because the constraints (32), (33) and (35) are *integro-differential equations*, the Hamiltonian for this maximization must – according to Hartl and Sethi (1984) or Grass et al. (2008, p. 423), and Bokov (2011) – be formulated as follows:[7]

$$\Delta(c,s,f,D,N,Z,m,K) \doteq \frac{1}{\rho+m}\Big\{u(c,s,f) + \lambda_N\big(\theta(D)\big)f\big(\theta(D)\big)N\big(\theta(D)\big) - \lambda_N mN - \lambda_Z mZ + \lambda_Z\big(\theta(D)\big)q(D)f\big(\theta(D)\big)N\big(\theta(D)\big) + (\gamma - s)\int_{\theta(t)}^{\theta(D)} \lambda_K(\xi)f(\xi)N(\xi)d\xi + \lambda_K[rK + w(1-bf)Z - (c+x+h+\gamma f)N] + \lambda_m v\big[M(P,D,h,g) - m\big]\Big\}, \tag{36}$$

where the co-state variables λ_m, λ_Z, λ_K and λ_m evolve according to

$$\frac{\dot\lambda_K}{\rho+m} = \frac{d\lambda_K}{d\theta} = \lambda_K - \frac{\partial\Delta}{\partial K} = \left(1 - \frac{r}{\rho+m}\right)\lambda_K \Leftrightarrow \frac{\dot\lambda_K}{\lambda_K} = \rho + m - r, \tag{37}$$

$$\frac{\dot\lambda_N}{\rho+m} = \frac{d\lambda_N}{d\theta} = \lambda_N - \frac{\partial\Delta}{\partial N} = \lambda_N + \frac{\lambda_N m + (c+x+h+sf)\lambda_K}{\rho+m} \Leftrightarrow$$

$$\dot\lambda_N = (\rho+2m)\lambda_N + (c+x+h+sf)\lambda_K, \tag{38}$$

$$\frac{\dot\lambda_Z}{\rho+m} = \frac{d\lambda_Z}{d\theta} = \lambda_Z - \frac{\partial\Delta}{\partial Z} = \left(1 + \frac{m}{\rho+m}\right)\lambda_Z - (1-bf)\frac{w}{\rho+m}\lambda_K$$

$$= \frac{1}{\rho+m}\big[(\rho+2m)\lambda_Z - (1-bf)w\lambda_K\big] =^{(6)} \frac{1}{\rho+m}\left[(\rho+2m)\lambda_Z - \frac{L}{Z}w\lambda_K\right]$$

$$\Leftrightarrow \dot\lambda_Z = (\rho+2m)\lambda_Z - w\frac{L}{Z}\lambda_K. \tag{39}$$

[7]Hartl and Sethi (1984) established this result for optimal control problems with a finite time horizon. Under the assumptions that the state variables (N, Z, K, m) are bounded over time, and that the functions $u(c, s, f)$ and $M(P, D)$, together with their partial derivatives, are bounded, Bokov subsequently extended this result to infinite-horizon problems. These conditions are satisfied in the present model.

$$\frac{\dot{\lambda}_m}{\rho+m} = \frac{d\lambda_m}{d\theta} = \lambda_m - \frac{\partial \Delta}{\partial m} = \lambda_m + \frac{1}{\rho+m}\big(\lambda_Z Z + \lambda_N N + \lambda_m v + \Delta\big)$$
$$\Leftrightarrow \quad \dot{\lambda}_m = (\rho+m+v)\lambda_m + \lambda_Z Z + \lambda_N N + \Delta. \tag{40}$$

The transversality conditions are given by
$\lim_{\theta\to\infty} \lambda_K(\theta)K(\theta)e^{-\theta} = 0,\ \lim_{\theta\to\infty} \lambda_N(\theta)N(\theta)e^{-\theta} = 0,$
$\lim_{\theta\to\infty} \lambda_Z(\theta)Z(\theta)e^{-\theta} = 0$ and $\lim_{\theta\to\infty} \lambda_m(\theta)m(\theta)e^{-\theta} = 0.$

A.4 Optimal controls

The household maximizes the Hamiltonian (36) by the control variables (c, s, f, h, D), yielding the following first-order conditions:[8]

$$0 = (\rho+m)\frac{\partial \Delta}{\partial c} = u_c - \lambda_K N \quad \Leftrightarrow \quad \lambda_K = u_c/N, \tag{41}$$

$$0 = (\rho+m)\frac{\partial \Delta}{\partial h} = v\lambda_m \frac{\partial M}{\partial h} - \lambda_K N \quad \Leftrightarrow \quad \lambda_m = \frac{N\lambda_K}{v\frac{\partial M}{\partial h}} =^{(41)} \frac{u_c}{v\frac{\partial M}{\partial h}}, \tag{42}$$

$$0 = (\rho+m)\frac{\partial \Delta}{\partial s} = u_s - \int_{\theta(t)}^{\theta(D)} \lambda_K(\xi)f(\xi)N(\xi)d\xi \quad \Leftrightarrow$$
$$u_s = \int_{\theta(t)}^{\theta(D)} \lambda_K(\xi)f(\xi)N(\xi)d\xi, \tag{43}$$

$$0 = (\rho+m)\frac{\partial \Delta}{\partial f} = u_f - (sN + bwZ)\lambda_K \quad \Leftrightarrow \quad s + wb\frac{Z}{N} = \frac{u_f}{N\lambda_K} =^{(41)} \frac{u_f}{u_c}, \tag{44}$$

$$\begin{aligned} 0 = (\rho+m)\frac{\partial \Delta}{\partial D} &= \lambda_Z\big(\theta(D)\big) f\big(\theta(D)\big) N\big(\theta(D)\big) q'(D) \\ &\quad + (\gamma - s)\lambda_K\big(\theta(D)\big) f\big(\theta(D)\big) N\big(\theta(D)\big)\theta'(D) \\ &=^{(30)} f\big(\theta(D)\big) N\big(\theta(D\big[\lambda_Z\big(\theta(D)\big)\big)q'(D) + (\gamma - s)\lambda_K\big(\theta(D)\big)(\rho+m)\big] \quad \Leftrightarrow \end{aligned}$$
$$s = \gamma + \frac{1}{\rho+m}\frac{\lambda_Z\big(\theta(D)\big)}{\lambda_K\big(\theta(D)\big)} q'(D). \tag{45}$$

The production function (7) is concave, while the utility function (15), the mortality driver $M(N, D, h, g)$ in (12) and the function (2) are strictly concave. Thus, the Hamiltonian (36) is strictly concave in the controls (c, s, f). Under these conditions, the equilibrium of the system (42)-(45) is unique.

[8]The superscript for "=" or "⇔" refers to the equation with the given number.

A.5 Steady state

In the household's system (32)-(35) and (37)-(45), there is a steady state where the control, state and co-state variables

$$(c, f, D, K, N, Z, m, \lambda_K, \lambda_N, \lambda_Z, \lambda_m)$$

are constants. In that steady state, it holds true that

$$\frac{dN}{d\theta} = 0 \;\Leftrightarrow^{(32)}\; f = m = M, \;\; \frac{dZ}{d\theta} = 0 \;\Leftrightarrow^{(33)}\; q = \frac{mZ}{fN} =^{f=m} \frac{Z}{N}, \tag{46}$$

$$\frac{d\lambda_K}{d\theta} = 0 \;\Leftrightarrow^{(37)}\; r = \rho + m, \tag{47}$$

$$\frac{dN}{d\theta} = \frac{dK}{d\theta} = \frac{dZ}{d\theta} = \frac{dm}{d\theta} = 0 \;\Leftrightarrow\; \Delta =^{(36)} \frac{u}{\rho + m} =^{(47)} \frac{u}{r}, \tag{48}$$

$$\begin{aligned}\frac{dK}{d\theta} = 0 \;\Leftrightarrow^{(35)}\; & rK + w(1 - bf)Z - (c + x + \gamma f)N \\ & = (s - \gamma)fN \int_{\theta(t-D)}^{\theta(t)} d\xi = (s - \gamma)fN \int_{\theta(t)}^{\theta(D)} d\xi,\end{aligned} \tag{49}$$

$$\Delta =^{(36), \frac{dN}{d\theta} = \frac{dZ}{d\theta} = \frac{dK}{d\theta} = 0} =^{(49)} \frac{u}{\rho + m} =^{(47)} \frac{u}{r}, \tag{50}$$

$$\frac{d\lambda_Z}{d\theta} = 0 \;\Leftrightarrow^{(39)}\; \frac{\lambda_Z}{\lambda_K} = \frac{wL/Z}{\rho + 2m} =^{(47)} \frac{wL/Z}{r + m} = \frac{N/Z}{r + m}\frac{wL}{N} =^{(46)} \frac{1/q}{r + m}\frac{wL}{N}, \tag{51}$$

$$\frac{d\lambda_N}{d\theta} = 0 \;\Leftrightarrow^{(38)}\; \frac{\lambda_N}{\lambda_K} = -\frac{c + x + h + sf}{\rho + 2m} =^{(46),(47)} -\frac{c + x + h + sm}{r + m}, \tag{52}$$

$$\begin{aligned}&\frac{d\lambda_m}{d\theta} = 0 \;\Leftrightarrow^{(40)} \\ &\frac{\lambda_m}{\lambda_K} = \frac{1}{\rho + m + v}\left(-\frac{\lambda_Z}{\lambda_K}Z - \frac{\lambda_N}{\lambda_K}N - \frac{\Delta}{\lambda_K}\right) \\ &\quad =^{(41),(47),(50),(51),(52)} \frac{1}{r + v}\left(-\frac{wL}{r + m} + \frac{c + x + h + sm}{r + m}N - \frac{N}{r}\frac{u}{u_c}\right) \\ &\quad = \frac{N}{r + v}\frac{1}{r + m}\left(c + x + h + sm - \frac{wL}{N} - \frac{r + m}{r}\frac{u}{u_c}\right).\end{aligned} \tag{53}$$

Inserting steady-state conditions (47)-(53) and definition (18) into the first-order conditions (42) and (42)-(45) yield

$$u_s = \lambda_K N m \theta(D) =^{(30),(41)} (\rho + m)u_c m D = rmDu_c, \tag{54}$$

$$s =^{(44)} \frac{u_f}{u_c} - bw\frac{Z}{N} =^{(46)} \frac{u_f}{u_c} - bwq, \tag{55}$$

$$s-\gamma =^{(45)} \frac{\lambda_Z q'}{(\rho+m)\lambda_K} =^{(47)} \frac{1}{r}\frac{\lambda_Z}{\lambda_K}q' =^{(46),(51)} \frac{1}{r}\frac{q'/q}{r+m}\frac{wL}{N} =^{(18)} z, \tag{56}$$

$$\frac{1}{\frac{\partial M}{\partial k}} = \frac{v}{N}\frac{\lambda_m}{\lambda_K} = \frac{1}{\frac{r}{v}+1}\frac{1}{r+m}\left(c+x+h+sm-\frac{wL}{N}-\frac{r+m}{r}\frac{u}{u_c}\right) \Leftrightarrow$$

$$\left(\frac{r}{v}+1\right)\frac{r+m}{\frac{\partial M}{\partial k}} = c+x+h+sm-\frac{wL}{N}-\frac{r+m}{r}\frac{u}{u_c}. \tag{57}$$

Equations (46) and (54)-(56) yield results (19)-(21).

B The social planner's optimum (eqs 22-25)

B.1 Transformation into virtual time

Using the transformation (29) in subsection A.1, the utility function (16) and the dynamics of the state variables – namely, the mass of workers (1), the mass of effective labor (3), capital (8) , and workers' mortality rate (12) with $P = N$ – can be reformulated from real time t into virtual time θ. Because the government's initial time is $t = 0$, the transformation yields the equations (31), (32), (33) and

$$\frac{dm}{d\theta} = \frac{v}{\rho+m}\big[M(N,D,h,g)-m\big]_, \quad m(0)=m_0, \tag{58}$$

$$\frac{dK}{d\theta} = \frac{1}{\rho+m}\left[F(K,L)-(c+h+g)N-s\int_{\theta(t-D)}^{\theta(t)} f(\xi)N(\xi)d\xi\right]_, \quad K(0)=K_0. \tag{59}$$

B.2 The Hamiltonian

The government maximizes utility (31) by choosing (c,s,f,h,g) and D from the set of constant controls subject to the dynamic constraints (32), (33), (58), (59). Since (32), (33) and (59) are integro-differential equations, the Hamiltonian for this problem must, according to Hartl and Sethi (1984) or Grass et al. (2008, p. 423), and Bokov (2011) – be formulated as follows:[9]

$$\Omega(E,c,s,f,D,N,Z,m) \doteq$$

[9] Hartl and Sethi (1984) established this result for optimal control problems with a finite time horizon. Under the assumptions that the state variables (N,Z,K,m) are bounded over time, and that the functions $u(c,s,f)$, $F(K,L)$ and $M(P,D)$, together with their partial derivatives, are bounded, Bokov subsequently extended this result to infinite-horizon problems. These conditions are satisfied in the present model.

$$
\frac{1}{\rho+m}\Big\{u(c,s,f)+\eta_N\big(\theta(D)\big)f\big(\theta(D)\big)N\big(\theta(D)\big)-\eta_N mN
+\eta_m v\big[M(\underbrace{N}_{P},D,E)-m\big]+\eta_Z\big(\theta(D)\big)q(D)f\big(\theta(D)\big)N\big(\theta(D)\big)-\eta_Z mZ
+\eta_K[F(K,\underbrace{(1-bf)Z}_{L})-(c+h+g)N]-s\int_{\theta(t)}^{\theta(D)}\eta_K(t)f(t)N(t)dt\Big\}, \tag{60}
$$

where the co-state variables η_N, η_K, η_Z and η_m evolve according to

$$
\frac{\dot{\eta}_K}{\rho+m}=\frac{d\eta_K}{d\theta}=\eta_K-\frac{\partial\Omega}{\partial K}=\left(1-\frac{F_K}{\rho+m}\right)\eta_K \;\Leftrightarrow\; \frac{\dot{\eta}_K}{\eta_K}=\rho+m-F_K, \tag{61}
$$

$$
\frac{\dot{\eta}_Z}{\rho+m}=\frac{d\eta_Z}{d\theta}=\eta_Z-\frac{\partial\Omega}{\partial Z}=\left(1+\frac{m}{\rho+m}\right)\eta_Z-(1-bf)\frac{F_L}{\rho+m}\eta_K
=\frac{1}{\rho+m}\big[(\rho+2m)\eta_Z-(1-bf)F_L\eta_K\big]=^{(6)}\frac{1}{\rho+m}\left[(\rho+2m)\eta_Z-\frac{L}{Z}F_L\eta_K\right]
$$

$$
\Leftrightarrow\;\dot{\eta}_Z=(\rho+2m)\eta_Z-F_L\frac{L}{Z}\eta_K, \tag{62}
$$

$$
\frac{\dot{\eta}_N}{\rho+m}=\frac{d\eta_N}{d\theta}=\eta_N-\frac{\partial\Omega}{\partial N}
=\eta_N+\frac{1}{\rho+m}\left[m\eta_N+(sf+c+h+g)\eta_K-v\frac{\partial M}{\partial P}\eta_m\right]\;\Leftrightarrow
$$

$$
\dot{\eta}_N=(\rho+2m)\eta_N+(sf+c+h+g)\eta_K-v\frac{\partial M}{\partial P}\eta_m, \tag{63}
$$

$$
\frac{\dot{\eta}_m}{\rho+m}=\frac{d\eta_m}{d\theta}=\eta_m-\frac{\partial\Omega}{\partial m}=\eta_m+\frac{1}{\rho+m}\big(\eta_Z Z+\eta_N N+\eta_m v+\Omega\big)\;\Leftrightarrow
$$

$$
\dot{\eta}_m=(\rho+m+v)\eta_m+\eta_Z Z+\eta_N N+\Omega. \tag{64}
$$

The transversality conditions are given by
$\lim_{\theta\to\infty}\eta_K(\theta)K(\theta)e^{-\theta}=0$, $\lim_{\theta\to\infty}\eta_N(\theta)N(\theta)e^{-\theta}=0$,
$\lim_{\theta\to\infty}\eta_Z(\theta)Z(\theta)e^{-\theta}=0$ and $\lim_{\theta\to\infty}\eta_m(\theta)m(\theta)e^{-\theta}=0$.

B.3 Optimal controls

The government's maximizes the Hamiltonian (60) with respect to control variables (c,s,f,D,h,g), yielding the following first-order conditions:[10]

$$
0=(\rho+m)\frac{\partial\Omega}{\partial c}=u_c-\eta_K N \;\Leftrightarrow\; u_c=\eta_K N \;\Leftrightarrow\; \eta_K=\frac{u_c}{N}, \tag{65}
$$

[10]The superscript for "=" or "⇔" refers to the equation with the given number.

$$0 = (\rho + m)\frac{\partial \Omega}{\partial h} = \eta_m v \frac{\partial M}{\partial h} - \eta_K N \;\;\Leftrightarrow\;\; \eta_m = \frac{\eta_K N}{v\frac{\partial M}{\partial h}} =^{(65)} \frac{u_c}{v\frac{\partial M}{\partial h}}, \tag{66}$$

$$0 = (\rho + m)\frac{\partial \Omega}{\partial g} = \eta_m v \frac{\partial M}{\partial g} - \eta_K N \;\;\Leftrightarrow\;\; \frac{\partial M}{\partial g} = \frac{\eta_K N}{\eta_m v} =^{(66)} \frac{\partial M}{\partial h}, \tag{67}$$

$$0 = (\rho + m)\frac{\partial \Omega}{\partial s} = u_s - \int_0^{\theta(D)} \eta_K(t) f(t) N(t) dt \;\;\Leftrightarrow$$

$$u_s = \int_0^{\theta(D)} \eta_K(t) f(t) N(t) dt, \tag{68}$$

$$0 = (\rho + m)\frac{\partial \Omega}{\partial f} = u_f - (F_L b Z + sN)\eta_K \;\;\Leftrightarrow\;\; s + F_L b \frac{Z}{N} = \frac{u_f}{N\eta_K} =^{(65)} \frac{u_f}{u_c}, \tag{69}$$

$$\begin{aligned}
0 = (\rho + m)\frac{\partial \Omega}{\partial D} &= \eta_Z\big(\theta(D)\big) f\big(\theta(D)\big) N\big(\theta(D)\big) q' \\
&\quad + \eta_m v \frac{\partial M}{\partial D} - s\eta_K(\theta(D)) f\big(\theta(D)\big) N\big(\theta(D)\big)\theta' \\
&=^{(30)} f\big(\theta(D)\big) N\big(\theta(D)\big) \Big\{ \eta_Z\big(\theta(D)\big) q' - s\eta_K\big(\theta(D)\big)[\rho + m(D)] \Big\} + \eta_m v \frac{\partial M}{\partial D} \\
\Leftrightarrow\;\; 0 &= \eta_Z\big(\theta(D)\big) q' - s\eta_K\big(\theta(D)\big)[\rho + m(D)] + \eta_m v \frac{\frac{\partial M}{\partial D}}{f\big(\theta(D)\big) N\big(\theta(D)\big)} \\
\Leftrightarrow\;\; s\eta_K\big(\theta(D)\big)[\rho + m(D)] &= \eta_Z\big(\theta(D)\big) q' + \eta_m v \frac{\frac{\partial M}{\partial D}}{f\big(\theta(D)\big) N\big(\theta(D)\big)} \\
\Leftrightarrow\;\; s &= \frac{1}{\rho + m(D)} \left[\frac{\eta_Z\big(\theta(D)\big)}{\eta_K\big(\theta(D)\big)} q'(D) + \frac{\eta_m v}{\eta_K\big(\theta(D)\big)} \frac{\frac{\partial M}{\partial D}}{f\big(\theta(D)\big) N\big(\theta(D)\big)} \right].
\end{aligned} \tag{70}$$

Because the production function (7) is concave and the utility function (15), the mortality driver $M(N, D, h, g)$ [cf., (12)] and the function (2) are strictly concave, the Hamiltonian (60) is strictly concave in controls (c, s, f, h) as well as in state variables (K, N, Z, m). Under these conditions, the equilibrium of the system (65)-(70) is unique.

B.4 Steady state

In the equilibrium of the system (59) and (61)-(70), the control variables (c, f, h, g, D), the state variables (K, L, m) and the co-state variables (η_K, η_Z, η_m) are constants, and it holds true that

$$\frac{dN}{d\theta} = \frac{dm}{d\theta} = 0 \;\;\Leftrightarrow^{(32),(58)}\;\; f = m = M, \tag{71}$$

$$\frac{dZ}{d\theta} = 0 \quad \Leftrightarrow^{(33),f=m} \ q = \frac{Z}{N}, \quad \frac{d\eta_K}{d\theta} = 0 \quad \Leftrightarrow^{(61)} \quad F_K = \rho + m, \tag{72}$$

$$u_s = \eta_K f N \theta(D) =^{(65),(71)} m u_c \theta(D) =^{(30)} m u_c (\rho + m) D =^{(72)} m u_c F_K D, \tag{73}$$

$$\frac{dN}{d\theta} = \frac{dK}{d\theta} = \frac{dZ}{d\theta} = \frac{dm}{d\theta} = 0 \quad \Leftrightarrow \quad \Omega =^{(60)} \frac{u}{\rho + m} =^{(72)} \frac{u}{F_K}, \tag{74}$$

$$\frac{d\eta_Z}{d\theta} = 0 \quad \Leftrightarrow^{(62)} \ \frac{\eta_Z}{\eta_K} = \frac{F_L L/Z}{\rho + 2m} =^{(72)} \frac{F_L L/Z}{F_K + m} =^{(72)} \frac{F_L L}{N} \frac{1/q}{F_K + m}, \tag{75}$$

$$\frac{d\eta_N}{d\theta} = 0 \quad \Leftrightarrow^{(63)} \quad \frac{\eta_N}{\eta_K} = \frac{1}{\rho + 2m} \left(v \frac{\partial M}{\partial P} \frac{\eta_m}{\eta_K} - c - h - g - sf \right)$$

$$=^{(72)} \frac{1}{F_K + m} \left(v \frac{\partial M}{\partial P} \frac{\eta_m}{\eta_K} - c - h - g - sm \right), \tag{76}$$

$$\frac{d\eta_m}{d\theta} = 0 \quad \Leftrightarrow^{(64)} \ \frac{\eta_m}{\eta_K} = \frac{1}{\rho + m + v} \left(-\frac{\eta_Z}{\eta_K} Z - \frac{\eta_N}{\eta_K} N - \frac{\Omega}{\eta_K} \right) \quad \Leftrightarrow^{(71),(72),(75),(76)}$$

$$\frac{\eta_m}{\eta_K} = \frac{1}{F_K + v} \left[-\frac{F_L L}{F_K + m} - \frac{N}{F_K + m} \left(v \frac{\partial M}{\partial P} \frac{\eta_m}{\eta_K} - c - h - g - sm \right) - \frac{\Omega}{\eta_K} \right]$$

$$=^{(65),(74)}$$

$$\frac{1}{F_K + v} \left[\frac{N}{F_K + m} \left(c + h + g + sm - v \frac{\partial M}{\partial P} \frac{\eta_m}{\eta_K} \right) - \frac{F_L L}{F_K + m} - \frac{u}{F_K} \frac{N}{u_c} \right]$$

$$= \frac{1}{F_K + v} \frac{N}{F_K + m} \left[c + h + g + sm - v \frac{\partial M}{\partial P} \frac{\eta_m}{\eta_K} - \frac{F_L L}{N} - \frac{F_K + m}{F_K} \frac{u}{u_c} \right]$$

$$\Leftrightarrow \quad \left[(F_K + v)(F_K + m) + N v \frac{\partial M}{\partial P} \right] \frac{\eta_m}{\eta_K}$$

$$= N \left(c + h + g + sm - \frac{F_L L}{N} - \frac{F_K + m}{F_K} \frac{u}{u_c} \right)$$

$$\Leftrightarrow \quad \left[\left(\frac{F_K}{v} + 1 \right) (F_K + m) + N \frac{\partial M}{\partial P} \right] \frac{v}{N} \frac{\eta_m}{\eta_K}$$

$$= c + h + g + sm - \frac{F_L L}{N} - \frac{F_K + m}{F_K} \frac{u}{u_c}. \tag{77}$$

B.5 The first best

Substituting steady-state conditions (72), (75) and (77) into the first-order conditions (66), (69) and (70) yields

$$\left[\left(\frac{F_K}{v} + 1 \right) (F_K + m) + N \frac{\partial M}{\partial P} \right] \frac{1}{\frac{\partial M}{\partial h}}$$

$$
\begin{aligned}
&=^{(66)} \left[\left(\frac{F_K}{v}+1\right)(F_K+m)+N\frac{\partial M}{\partial P}\right]\frac{v}{N}\frac{\eta_m}{\eta_K} \\
&= c+h+g+sm-\frac{F_L L}{N}-\frac{F_K+m}{F_K}\frac{u}{u_c}, \qquad (78)
\end{aligned}
$$

$$
s =^{(69)} \frac{u_f}{u_c}-F_L b\frac{Z}{N} =^{(72)} \frac{u_f}{u_c}-F_L bq, \qquad (79)
$$

$$
\begin{aligned}
s &=^{(70)} \frac{1}{\rho+m}\left(\frac{\eta_Z}{\eta_K}q'+\frac{v}{N}\frac{\eta_m}{\eta_K}\frac{1}{m}\frac{\partial M}{\partial D}\right) =^{(72)} \frac{1}{F_K}\left(\frac{\eta_Z}{\eta_K}q'+\frac{v}{N}\frac{\eta_m}{\eta_K}\frac{1}{m}\frac{\partial M}{\partial D}\right) \\
&=^{(66),(75)} \frac{1}{F_K}\left(\frac{F_L L}{N}\frac{q'/q}{F_K+m}+\frac{1}{m}\frac{\partial M}{\partial D}\Big/\frac{\partial M}{\partial h}\right). \qquad (80)
\end{aligned}
$$

Results (72), (79) and (80) yield (22)-(25).

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