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Optimal Education Policy, R&D Productivity, and Endogenous Growth with Heterogeneous Individuals

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Abstract

This study characterizes optimal policy in an endogenous growth model where heterogeneous individuals choose between entering the workforce as unskilled labor or pursuing education to become skilled. In the decentralized economy, monopolists produce goods using both labor types. They purchase innovations from R&D firms, which employ skilled labor whose productivity increases with education duration. We identify two fundamental market distortions: the extensive margin (the decision to acquire skills) and the intensive margin (the duration of schooling). The social planner corrects these distortions through two primary instruments. A student grant is employed to internalize the positive externalities of education. Conversely, a targeted income tax on skilled labor corrects the intensive margin; by reducing the marginal benefit of prolonged schooling, the tax mitigates over-education and ensures a timely transition into the productive workforce. The government budget is balanced via a consumption tax, while government-backed loans relax liquidity constraints for heterogeneous students. The justification for these interventions vanishes when economic growth is absent or R&D productivity is independent of education duration. In these limiting cases, a laissez-faire approach is socially optimal, underscoring that education policy must be dynamically linked to the broader macroeconomic innovation environment.

Journal of Economic Literature: O41; I25; H21; J24

Keywords: Endogenous growth, Human capital, R&D productivity, Optimal taxation, Student grants, Individual heterogeneity, Liquidity constraints

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1 Introduction

This study examines the design of optimal education policy within a framework of R&D-driven endogenous growth. We model an economy where individuals with heterogeneous abilities must choose between immediate entry into the labor market as unskilled workers or investing in education to become skilled. In our framework, skilled labor serves as the critical engine for production, education, and R&D. Our analysis is motivated by a fundamental externality: expanding the scale and duration of education accelerates technological progress and, by extension, aggregate economic growth. Consequently, the government must strategically use taxation and subsidies to align individual occupational choices with the social optimum.

Empirical evidence consistently supports the link between education duration and positive growth externalities. Agiomirgianakis et al. (2002) demonstrate across a 93-country panel that while all educational tiers promote growth, higher education provides the most substantial impetus. This is echoed by Gyimah-Brempong et al. (2006), who identify a significant growth elasticity with respect to higher education in African economies. Furthermore, national case studies ranging from Nigeria (Lawal and Wahab, 2011) to the OECD (Yardimcioglu et al., 2014) confirm a robust, long-term cointegration between educational investment and economic performance.

Building on these findings, we posit that education duration correlates with growth because skilled workers – particularly those in R&D and management – generate knowledge spillovers that enhance innovation productivity. Unlike models with exogenous human capital, we treat both enrollment and study duration as endogenous choices influenced by policy instruments such as student grants and income taxes. Our research integrates and extends the prevailing growth literature in several key dimensions as follows.

While Greiner and Semmler (2002) use a Romer-style model to link education to capital externalities, we adopt a framework where monopolistic producers acquire patents from a specialized R&D sector (cf. Peretto and Connolly, 2007), endogenizing education duration rather than treating it as a fixed requirement. Grossmann (2007) highlights the superiority of public education funding over R&D subsidies in OLG models. We depart from this by assuming individuals possess private information regarding their own abilities. This information asymmetry necessitates that the government influence attainment indirectly through the tax system rather than via direct mandates. Unlike Hori and Yamada (2013), who treat human capital as an accumulating asset, we model education as a process that enhances worker efficiency, allowing for a specific focus on the marginal returns of study duration. We extend the stationary population framework of Findlay and Kierzkowski (1983) and Borsook (1987) by assuming that a worker's ultimate efficiency is endogenously determined by their time investment, rather than being exogenous to their innate ability.

Our first contribution is the identification of the boundary conditions for educational intervention. We establish that the rationale for public policy is strictly contingent on the presence of growth-generating externalities. Specifically, the justification for government intervention – via both student grants and corrective taxes – vanishes in two limiting cases: a stagnant economy or a scenario where R&D productivity is orthogonal to education duration. In these instances, private and social returns to schooling align perfectly, rendering laissez-faire optimal. This result provides a theoretical bridge between growth-oriented policy and traditional human capital models. Our second contribution is the construction of a policy suite that maintains the social optimum when growth is driven by two distict forces: profit-maximizing R&D firms and households investing in heterogeneous human capital.

The remainder of the study is organized as follows: Section 2 analyzes labor supply and educational choice. Section 3 details the role of public policy in the student experience. Section 4 explores the behavior of final-goods firms, intermediate monopolists, and the R&D sector. Section 5 examines intertemporal household behavior. Section 6 characterizes the social optimum and derives the optimal policy suite by reconciling planner benchmarks with private equilibrium conditions. Finally, Section 7 concludes.

2 Abilities, education and labor supply

2.1 Fertility and mortality

In the economy, time t is continuous and there is a continuum $j \in [0,1]$ of *individuals*. At any time t, an exogenous mass $m \subset [0,1)$ of new individuals is born. Simultaneously, the same mass m of old individuals die out. In fertility-choice models of population growth, it is commonly assumed – primarily for analytical tractability – that all individuals are subject to a uniform mortality rate.[1] In other words, the instantaneous probability of death is assumed to be independent of age. We adopt the same simplifying assumption. We assume furthermore – again for analytical tractability – that the mortality rate equals the fertility rate m.[2] Then, the population size remains unity and all aggregate variables can be interpreted as per capita quantities.

2.2 Heterogeneous individuals

Following Borsook (1987) and Findlay and Kierzkowski (1983), we assume that newborn $j \in [0, m]$ either stays as a worker or studies to fully exploit their specific, but exogenous abilities.[3] In this document, we extend that setup so that the students can raise their forthcoming efficiency above their exogenous abilities by lengthening their education.

The newborns $j \in [0, m]$ differ in their *exogenous natural ability* $\kappa(j)$ as a skilled worker. In the model, we organize them so that higher j corresponds to higher natural ability, $\kappa'(j) > 0$. The distribution of natural ability remains the same for individuals born at any point in time. Let D denote education duration. An individual born at time t can raise their future productivity as a skilled worker, denoted by q, by undertaking education from t to $t + D$. The acquired education increases the individual's efficiency according to a continuous, increasing, and concave function $q(D)$. Given the exogenous mortality rate m, the probability of surviving the education period of length D is e^{-mD}. Consequently, the individual's supply of effective

[1] See Razin and Ben-Zion (1975), Becker (1981), Galor and Weil (1996), Stelter (2016), Palokangas (2021), and Lehmijoki and Palokangas (2023).

[2] Following the family-optimization framework established by Razin and Ben-Zion (1975), Becker (1981), and others (Galor and Weil, 1996; Stelter, 2016; Palokangas, 2021), we can endogenize the fertility rate. Similarly, the mortality rate can be endogenized using the approach of Lehmijoki and Palokangas (2023). Because these extensions maintain equal fertility and mortality rates in the steady state, the resulting policy rules remain fundamentally consistent with those derived in this study.

[3] Borsook (1987) and Findlay and Kierzkowski (1983) ignore the period of elementary education, for simplicity.

skilled labor at time $t + D$ will be

$$e^{-mD}[\kappa(j) + q(D)], \quad \kappa' > 0, \quad q(0) = 0, \quad q' > 0, \quad q'' < 0. \tag{1}$$

2.3 Supply of unskilled labor

Since newborns $j \in [0, m]$ are heterogeneous in their natural ability $\kappa(j)$, there exists a *threshold worker* $b \in [0, m]$ who is indifferent between pursuing education and entering the workforce. This threshold partitions the population: individuals with $j \in [0, m]$ remain as unskilled laborers, providing an aggregate labor supply of $\int_0^b dj = b$, whereas those with $j \in [b, m]$ study to become skilled workers.

Individuals born at time t provide a flow of new unskilled workers $b(t)$. Meanwhile, the existing unskilled labor force l faces a constant mortality rate m. The dynamics of the unskilled labor supply at time t are therefore governed by the net difference between the entry of new workers and the exit of the deceased:

$$\dot{l}(t) \doteq \frac{dl}{dt}(t) = b(t) - ml(t), \quad l(0) = l_0, \tag{2}$$

where l_0 is the initial mass of unskilled labor at time 0.

2.4 Supply of effective skilled labor

Summing the contributions (1) across all students $j \in [b, m]$ indicates that the individuals born at time t supply, after education duration D, at time $t + D$, the following mass of skilled workers' effective labor;

$$\begin{aligned}
h(b, D) &\doteq \int_b^m e^{-mD}[\kappa(j) + q(D)]dj = e^{-mD} \int_b^m [\kappa(j) + q(D)]dj > 0, \\
h_D &\doteq \frac{\partial h}{\partial D} = e^{-mD} q' \int_b^m dj - mh = \underbrace{e^{-mD}}_{+} (\underbrace{m - b}_{+}) \underbrace{q'}_{+} - \underbrace{mh(b, D)}_{+}, \\
h_b &\doteq \frac{\partial h}{\partial b} = -e^{-mD}[\kappa(b) + q(D)] < 0, \quad \frac{\partial^2 h}{\partial b^2} = -\underbrace{e^{-mD}}_{+} \underbrace{\kappa'}_{+} < 0, \\
\frac{\partial^2 h}{\partial D^2} &= (\underbrace{m - b}_{+}) \underbrace{q''}_{-} - m \underbrace{h_D}_{-}, \quad \frac{\partial^2 h}{\partial b \partial D} = -\underbrace{m}_{+} \underbrace{h_b}_{-} - \underbrace{e^{-mD}}_{+} \underbrace{q'}_{+},
\end{aligned} \tag{3}$$

where h_D represents the *marginal effect of education* D *on effective skilled labor* and $|h_b| = -h_b$ *a threshold worker's effective skilled labor.*[4] To enable a unique equilibrium for the individuals, we assume that the supply of skilled workers' effective labor (3) is a concave function.

Given an education duration D, the inflow of effective skilled labor at time t is determined by the cohort born at time $t-D$. Simultaneously, the existing stock of skilled workers faces a constant mortality rate m. The dynamics of the skilled labor supply s are thus governed by the net difference between the arrival of new skilled workers, (3), and the reduction in the skilled labor mass due to mortality:

$$\dot{s}(t) \doteq \frac{ds}{dt}(t) = h(b(t-D), D) - ms(t), \quad s(0) = s_0, \tag{4}$$

where s_0 is the initial mass of effective skilled labor at time 0 and $b(t-D)$ the value of variable b at time $t-D$.

3 Students and public policy

Since the individuals $m - b(\xi)$ having started their studies at times $\xi \in (t-D, t]$ are still students at time t, the mass of students at time t is

$$\int_{t-D}^{t} [m - b(\xi)] d\xi. \tag{5}$$

We assume that the *educators* are skilled workers,[5] and that each of the students (5) requires a constant amount α of the educators' input. Consequently, the total education costs in terms of effective skilled labor are

$$\alpha \int_{t-D}^{t} [m - b(\xi)] d\xi. \tag{6}$$

With no money in the model, we choose unskilled labor l as the *numeraire*, normalizing its wage at unity. Let w denote the relative wage for effective

[4] When a worker transitions from unskilled labor to skilled labor, their contribution as a skilled worker is represented by $-h_b > 0$.

[5] The mathematical method being used in this document requires that lags are additive in the integro-differential equations [Cf., Hartl and Sethi (1984) or Grass et al. (2008), p. 423]. If the education costs (6) were defined in terms of the consumption good, that requirement would not be feasible.

skilled labor s. Multiplying education costs in terms of skilled labor, (6), by the relative wage w yields education costs in terms of the *numeraire*:

$$w\alpha \int_{t-D}^{t} [m - b(\xi)] d\xi. \tag{7}$$

The government provides a *grant* g for each of the students (7), proportional to the skilled worker's wage w. Additionally, it imposes a tax x on net skilled labor income. This income is defined as total skilled wages ws minus education costs net of grants,[6] $(\alpha - g)w \int_{t-D}^{t}[m - b(\xi)]d\xi$.

To maintain tractability in a model of endogenous growth despite individual heterogeneity, we omit idiosyncratic lump-sum taxes and assume a balanced government budget funded by a tax φ on consumption expenditures pc. Here, p denotes the price of consumption c relative to the *numeraire* (unskilled labor). Consequently, the government budget constraint is

$$gw \int_{t-D}^{t} [m - b(\xi)] d\xi = x \left\{ ws(t) + (g - \alpha) w \int_{t-D}^{t} [m - b(\xi)] d\xi \right\} + \varphi pc. \tag{8}$$

4 Firms

4.1 Competitive firms

The quantity of consumption c is produced from the quantities of the continuum of intermediate goods $i \in [0, 1]$ with symmetric CES technology:

$$c = y \doteq \left(\int_0^1 y_i^{1-1/\sigma} di \right)^{\sigma/(\sigma-1)} \quad \text{with } \sigma > 1, \tag{9}$$

where y_i is the quantity of good $i \in [0, 1]$ and σ the constant *elasticity of substitution* between any pair of the inputs $i \in [0, 1]$.

Competitive final-good producers maximize their profits by their inputs y_i, given the output price p and the input prices p_i, $i \in [0, 1]$. Because technology (9) exhibits constant returns to scale, in equilibrium, they earn no profits and set the input prices p_i equal to the marginal products $p\frac{\partial y}{\partial y_i}$:

$$p_i = p\frac{\partial y}{\partial y_i} = p\left(\frac{y}{y_i}\right)^{1/\sigma} \Leftrightarrow \; y_i = y\left(\frac{p}{p_i}\right)^{\sigma} \text{ with } \frac{y_i}{p_i}\frac{\partial p_i}{\partial y_i} = -\frac{1}{\sigma}. \tag{10}$$

[6] To derive more tractable optimal tax rules, net education costs are deducted from skilled labor income ws within the tax base.

4.2 Monopolists and R&D firms

To avoid excessive complications in the model, we assume that both monopolists and R&D firms are risk-neutral. Each intermediate good $i \in [0,1]$ is produced on a dedicated production line by a single monopolist with the same label i. The efficiency of monopolist i, denoted by a_i, evolves according to a quality ladder framework, where accumulated knowledge from past R&D enhances the productivity of current research. Innovations for each monopolist are developed by independent, competitive R&D firms that sell their patented findings to the monopolists.[7]

Units are defined such that, during an infinitesimal time interval dt, the allocation of one unit of effective skilled labor in R&D yields a patented innovation with probability $\delta\, dt$. The parameter $\delta > 0$ represents *R&D efficiency*. Each successful innovation upgrades the monopolist's efficiency from the prevailing state-of-the-art level a to a new level a_i.

To endogenize technological change, we model the process as an extensive-form game featuring a stochastic player. The game unfolds as follows:

Stage 1: R&D innovation. R&D firm $i \in [0,1]$ employs effective skilled labor v_i to develop a patented innovation. With probability $\delta v_i dt$, the firm successfully raises efficiency to a_i, with probability $1 - \delta v_i dt$, the attempt fails and efficiency remains at a. This stochastic process is described by

$$\frac{da_i}{a} = \delta v_i dt. \tag{11}$$

Stage 2: knowledge diffusion. The stochastic player distributes the innovations of any R&D firm i to all other firms $\varsigma \in [0,1] \setminus i$.

Stage 3: patent acquisition. Monopolist i acquires patents z_i from the R&D firm specialized in its production line. This allows the monopolist to enhance its productivity from the frontier a to its specific level a_i.

Stage 4: intermediate production. Monopolist i produces quantity y_i of intermediate good i using unskilled labor l_i and effective skilled labor n_i,

[7]For a similar approach, see Peretto and Connolly (2007).

according to a concave, linearly homogeneous function f:

$$y_i = a_i f(l_i, n_j), \quad f_l \doteq \frac{\partial f}{\partial l_i} > 0, \quad f_n \doteq \frac{\partial f}{\partial n_i} > 0, \quad f_{ln} \doteq \frac{\partial f^2}{\partial l_i \partial n_i} > 0. \tag{12}$$

We solve this game using backward induction in the following analysis.

4.2.1 Intermediate production

Monopolist i maximizes its *operating profit* $\Pi_i \doteq p_i y_i - l_i - w n_j$ by choosing inputs l_i and n_i. This maximization is subject to the production technology (12) and the output demand curve (10), taking as given the efficiency parameter a_i the skilled labor wage w, and the final good's price p and quantity y. This defines the function

$$\begin{aligned}\Pi_i = \Pi(a_i, w, p, y) &\doteq \max_{(l_i, n_i) \text{ s.t. } (10),(12)} (p_i y_i - l_i - w n_j) \\ &= \max_{l_i, n_i} \left[p y^{1/\sigma} a_i^{1-1/\sigma} f(l_i, n_j)^{1-1/\sigma} - l_i - w n_j \right]. \end{aligned} \tag{13}$$

The maximization in (13) yields the first-order conditions

$$\frac{\partial \Pi_i}{\partial l_i} = 0 \;\Leftrightarrow\; 1 = \left(1 - \frac{1}{\sigma}\right) p y^{1/\sigma} a_i^{1-1/\sigma} \frac{f_l(l_i, n_j)}{f(l_i, n_j)^{1/\sigma}}, \tag{14}$$

$$\frac{\partial \Pi_i}{\partial n_i} = 0 \;\Leftrightarrow\; w = \left(1 - \frac{1}{\sigma}\right) p y^{1/\sigma} a_i^{1-1/\sigma} \frac{f_n(l_i, n_j)}{f(l_i, n_j)^{1/\sigma}} \;\Leftrightarrow\; \frac{f_n}{f_l} = w. \tag{15}$$

By (13) and (14), the marginal revenue of efficiency for monopolist i is

$$\frac{\partial \Pi}{\partial a_i} = \left(1 - \frac{1}{\sigma}\right) p y^{1/\sigma} a_i^{-1/\sigma} f^{1-1/\sigma} = \underbrace{\left(1 - \frac{1}{\sigma}\right) p y^{1/\sigma} a_i^{1-1/\sigma} f^{-1/\sigma}}_{1/f_l} \frac{f}{a} = \frac{f}{a f_l}. \tag{16}$$

Inserting the first-order conditions (14) and (15) back into (13) and noting the linear homogeneity of the function (12) shows that the operating profit Π_i is in fixed proportion $\frac{1}{\sigma}$ to the value of output, $p_i y_i$:

$$\Pi_i = p_i y_i - l_i - w n_j = p_i a_i \left[f - (1 - 1/\sigma)(\underbrace{l_i f_l + n_i f_n}_{=f}) \right] = \frac{p_i a_i f}{\sigma} = \frac{p_i y_i}{\sigma}. \tag{17}$$

4.2.2 Patent acquisition

Because during an infinitesimal time interval dt, the allocation of one unit of effective skilled labor in R&D yields one patented innovation with probability $\delta\, dt$ and fails to do so with probability $1 - \delta\, dt$, then, the expected amount of patents is $z_i = \delta v_i$, where v_i is labor employed in R&D. By this and (11), efficiency in the production of good i evolves as follows:

$$\dot{a}_i \doteq \frac{da_i}{dt} = \delta v_i a = z_i a, \quad z_i \geq 0. \tag{18}$$

Let P_i represent the price of a patent z_i specifically designed for production line $i \in [0, 1]$. Monopolist i earns the expected total profit

$$\Theta_i \doteq \Pi_i - P_i z_i, \tag{19}$$

where Π_i is its operating profit (13) and $P_i z_i$ its costs of purchasing patents z_i. Monopolist $i \in [0, 1]$ maximizes its expected present value

$$\int_0^\infty \Theta_i e^{-rt} dt \tag{20}$$

subject to the evolution of its efficiency a_i, (18), by its demand for patents z_i, given the prevailing interest rate r, the skilled labor wage w, the price p and quantity y of the final good, the state-of-the-art efficiency a and its specific patent price P_i. This yields the condition (cf., Appendix A)

$$P_i = \frac{a}{r + \gamma} \frac{\partial \Pi}{\partial a_i}, \tag{21}$$

where $\gamma \doteq \frac{\dot{a}}{a}$ is the *productivity growth rate*. Accordingly, the unit cost of a patent – equating to the skilled wage w – must match the monopolist's marginal revenue from an increased efficiency a_i, $\frac{\partial \Pi}{\partial a_i}$, scaled by the R&D productivity a. This marginal benefit is discounted at the effective discount rate $r + \gamma$ which accounts for both the interest rate r and the economy's growth-driven obsolescence γ.

4.2.3 Knowledge diffusion

The stochastic player moves to distribute the successful innovations. This updates the global knowledge pool. The state-of-the-art a shifts upward to

$$a \doteq \max_{\varsigma \in [0,1]} a_\varsigma, \tag{22}$$

which then becomes the baseline for the next "round" of the ladder.

Given that the probability of a successful innovation within an infinitesimal interval dt is small [cf. (11)], the likelihood of two or more monopolists innovating simultaneously is negligible. Consequently, the probability that the state-of-the-art productivity (22) increases in the interval dt is

$$\begin{aligned}\frac{da}{a} &= \lim_{dt\to 0}\left[\int_0^1 \frac{da_i}{a}di + \int_{i\in[0,1]}\int_{\varsigma\in[0,1]} \frac{da_i}{a}\frac{da_\varsigma}{a}di\,d\varsigma\right] \\ &= \delta\,dt\,\lim_{dt\to 0}\Bigg[\underbrace{\int_0^1 v_i di}_{=v} + \underbrace{\left(\int_{i\in[0,1]} v_i di \int_{\varsigma\in[0,1]} v_\varsigma d\varsigma\right)dt}_{\to 0}\Bigg] = \delta v\,dt,\end{aligned}$$

where $v \doteq \int_0^1 v_i di$ is total R&D input in the economy. This determines the productivity growth rate

$$\gamma \doteq \frac{\dot{a}}{a} \doteq \frac{1}{a}\frac{da}{dt} = \delta v, \quad v \geq 0, \quad a(0) = a_0, \tag{23}$$

where a_0 is the initial of efficiency a at time 0.

4.2.4 R&D investment

From (11), it follows that during an infinitesimal time interval dt, the R&D firm earns expected profit

$$\mathcal{R}_i \doteq P_i\delta v_i dt - wv_i\,dt = (P_i\delta - w)v_i dt, \tag{24}$$

where $P_i\delta v_i dt$ represents expected revenue and $wv_i dt$ denotes expenditures on effective skilled labor v_i. Assuming free entry into the R&D sector, expected profits (24) must equal zero in equilibrium, implying that the patent price is

$$P_i = w/\delta. \tag{25}$$

4.3 The general equilibrium of the firms

Because the system (9)-(23) is symmetric over $i \in [0,1]$, in equilibrium it holds true that $l_i = l$, $n_i = n$, $s_i = s$, $a_i = a$, $y_i = y$, $p_i = p$, $\Pi_i = \Pi$, $\Theta_i = \Theta$, $v_i = v$ and $P_i = P$. Then, from equations (9), (12) and (14) it follows that

$$c = y = af(l,n), \tag{26}$$

$$\sigma/(\sigma-1) = py^{1/\sigma}a^{1-1/\sigma}f^{-1/\sigma}f_l = p(af)^{1/\sigma}a^{1-1/\sigma}f^{-1/\sigma}f_l = paf_l(l,n). \tag{27}$$

Effective skilled labor s is used in production n, R&D v and education (6):

$$s = n + v + \alpha \int_{t-D}^{t} [m - b(\xi)] d\xi. \tag{28}$$

From (15), (16), (21), (23) and (25) it follows that

$$\frac{f_n}{f_l} = w = \delta P_i = \frac{a\delta}{r+\gamma} \frac{\partial \Pi}{\partial a_i} = \frac{\delta(D)}{r+\gamma} \frac{f}{l_l} \quad \Leftrightarrow \quad \frac{f_n}{f} = \frac{1}{r+\gamma} = \frac{1}{r+\delta v}. \tag{29}$$

5 Cohorts

5.1 Utility

Households supply labor, own all firms, and consume the final good. They allocate savings between education and loans to monopolists for patent investment. To address the liquidity constraints arising from individual heterogeneity in education financing, we assume a government guarantee on student loans. This policy ensures that any individual j, regardless of ability $\kappa(j)$, can access the capital market at the prevailing interest rate r. Such symmetry eliminates credit frictions, allowing the analysis to focus on a *representative cohort* – specifically, individuals born at $t = 0$.

We assume that the representative individual's instantaneous utility at time t is the logarithm of their consumption, $\ln c(t)$. Given a constant mortality rate m, the probability of surviving from time 0 until time ξ is $e^{-m\xi}$, making the expected instantaneous utility at that time $e^{-m\xi} \ln c(\xi)$. Consequently, the individual's expected intertemporal utility at $t = 0$ is the integral of these instantaneous utilities over the infinite horizon $[0, \infty)$, discounted at the constant rate of time preference $\varrho > 0$:

$$U \doteq \int_0^\infty e^{-m\xi} [\ln c(\xi)] e^{-\varrho\xi} d\xi = \int_0^\infty [\ln c(\xi)] e^{-\rho\xi} d\xi, \tag{30}$$

where $\rho \doteq \varrho + m$ is the *effective rate of time preference* with mortality.

5.2 Saving

The cohort's asset portfolio consists of unskilled labor l, skilled labor s, and wealth W, the latter of which is denominated in terms of the *numeraire* (i.e., unskilled labor). The cohort takes the interest rate r, monopolist profits Θ and the consumption price p as given. Furthermore, it treats the following as exogenous: the relative wage for effective skilled labor w, the student grant g, the consumption tax ϕ, and the tax x levied on skilled labor income net of education costs, $ws + (g - \alpha)w \int_{t-D}^{t} [m - b(\xi)]d\xi$. Consequently, at any time t, the cohort's wealth W evolves according to

$$\dot{W} = rW + \Theta + l + (1 - x)\left\{ ws + (g - \alpha)w \int_{t-D}^{t} [m - b(\xi)]d\xi \right\} - (1 + \varphi)pc. \tag{31}$$

5.3 Optimal behavior

At time $t = 0$, the cohort chooses its consumption path $c(\xi)$ for t $\xi \in [0, \infty)$. Furthermore, since all its study plans must be determined at the initial date $t = 0$, it selects the mass of its members who remain unskilled workers, $b \in [0, m]$, and the duration of education D for its students $m - b$ from the *set of constant controls*. The cohort maximizes its intertemporal utility (30) subject to the asset evolution equations (2), (4), and (31), taking the paths of $(\Theta, r, p, w, g, \varphi)$ as given.

5.4 Equilibrium conditions

The cohort's equilibrium conditions are derived in Appendix B as follows:

$$r = \rho, \tag{32}$$

$$\frac{h_D}{r + m} = (\alpha - g)(m - b), \tag{33}$$

$$\frac{1}{r + m} \frac{1}{(1 - x)w} = (g - \alpha)D + \frac{|h_b|}{r + m}. \tag{34}$$

Equation (32) yields the standard result that the interest rate r equals the rate of time preference ρ.

Since unskilled labor l and effective skilled labor s are assets that depreciate at the mortality rate m, their respective income streams are discounted at the *effective discount rate* – the sum of the interest rate r and the depreciation rate m. The term $\alpha - g$ denotes the net cost per student (gross cost α less the grant g), while $m - b$ is the marginal mass of students.

According to (33), the marginal return of education duration D on skilled labor productivity, h_D, discounted at the effective discount rate $r+m$, must equal the marginal net cost of that investment, $(\alpha - g)(m - b)$. Similarly, (34) establishes that an unskilled worker's wage flow (normalized to unity), discounted at the effective discount rate $r+m$ and adjusted by the after-tax relative wage of skilled labor $(1-x)w$, must equal the sum of the student's total net education cost, $(\alpha - g)D$, and the threshold worker's effective skilled labor contribution, $|h_b|$, also discounted at the effective discount rate $r+m$.

6 The government

Based on comprehensive evidence linking economic growth to education duration (cf., Section 1), we postulate that an increase in the duration of R&D workers' education, D, promotes R&D firms' success in innovation, δ:

$$\delta(D), \quad \delta' > 0, \quad \varepsilon(D) \doteq \frac{\delta'(D)}{\delta(D)} D > 0, \tag{35}$$

where ε is the *elasticity of the efficiency of R&D*, δ, with respect to education duration D.

To derive the first-best solution, we consider the government as a social planner that maximizes the cohort's utility (30) by allocating skilled labor between production n and R&D v, and by choosing the contribution to unskilled workers, b, and the duration of education D from the set of constant controls.[8] This optimization is subject to technological change (23), the asset evolution equations of unskilled and skilled labor, (2) and (4), and the market-clearing condition for skilled labor (28). The solution yields the first-best conditions (see Appendix C).

$$\frac{h_D}{r+m} = (m-b)\alpha - \frac{f}{f_n}\frac{1}{r}\delta' v, \quad \frac{|h_b|}{r+m} = \alpha D + \frac{1}{r+m}\frac{f_l}{f_n}. \tag{36}$$

Noting firms' equilibrium conditions (15) and (29), the growth rate (23) and

[8] By Walras' Law, the government budget constraint (8) is redundant, as it is implicitly satisfied through the linear combination of the budget constraints of cohorts and and the market-clearing conditions.

the definition of elasticity ε in (35), the results (36) can be written as follows:

$$\frac{h_D}{r+m} = (m-b)\alpha - \frac{\delta v + r}{r}\frac{\varepsilon}{D}v, \tag{37}$$

$$\frac{|h_b|}{r+m} = \alpha D + \frac{1}{r+m}\frac{1}{w}. \tag{38}$$

If the government could directly determine education duration D, then result (37) can be interpreted as follows:

Proposition 1 *The marginal contribution of education duration to effective skilled labor, h_D, discounted at the effective discount rate $r+m$, $\frac{h_D}{r+m}$, must be set equal to the marginal cost of education length, $(m-b)\alpha$, minus the marginal effect of education duration through R&D efficiency, $\frac{\delta v+r}{r}\frac{\varepsilon}{D}v$: $\frac{h_D}{r+m} = (m-b)\alpha - \frac{\delta v+r}{r}\frac{\varepsilon}{D}v$.*

If the government could directly select the individuals b who should remain unskilled, then the result (38) can be interpreted as follows:

Proposition 2 *The demand for educators during the study period, αD, combined with a threshold worker's input as a skilled worker, $|h_b| = -h_b$, discounted at the effective discount rate $r+m$, must be set equal to the inverse of the relative wage for skilled labor, $\frac{1}{w}$, discounted by the same rate: $\alpha D + \frac{|h_b|}{r+m} = \frac{1}{r+m}\frac{1}{w}$.*

The first-best policy derived from Propositions 1 and 2 requires comprehensive information regarding individual abilities $\kappa(j)$ for all $j \in [0,1]$, and thus the exact functional form of (3). Given that such granular data may be unavailable to the policymaker, direct implementation may be infeasible. Consequently, we turn to a decentralized policy framework where the targets (b, D) are influenced indirectly via the tax system.

Substituting the government's equilibrium condition (37) into the cohort's equilibrium condition (33), we obtain

$$(\alpha - g)(m-b) = \frac{h_D}{r+m} = (m-b)\alpha - \frac{\delta v + r}{r}\frac{\varepsilon}{D}v \;\Leftrightarrow$$

$$-g(m-b) = -\frac{\delta v + r}{r}\frac{\varepsilon}{D}v \;\Leftrightarrow\; g = \frac{\delta v/r + 1}{(m-b)D}\varepsilon v. \tag{39}$$

Comparing the equilibrium conditions for the cohort (34) and the government (38) yields the optimal tax rate on skilled labor:

$$(r+m)(\alpha-g)D+\frac{1}{(1-x)w}=|h_b|=(r+m)\alpha D+\frac{f_l}{f_n}=(r+m)\alpha D+\frac{1}{w}$$

$$\Leftrightarrow\quad \frac{x}{1-x}=\frac{1}{1-x}-1=(r+m)wgD \quad\Leftrightarrow\quad x=\left[1+\frac{1}{(r+m)wgD}\right]^{-1}<1. \tag{40}$$

The policy rules (39) and (40) imply that as $\varepsilon v \to 0$, both g and x vanish: $\lim_{(\varepsilon v)\to 0} g = 0$ and $\lim_{(\varepsilon v)\to 0} x = \lim_{g\to 0} x = 0$. These limiting cases can be characterized as follows:

Proposition 3 *If economic growth is absent (i.e., if $v \to 0$) or if R&D productivity is independent of education duration (i.e., if $\varepsilon \to 0$), then the rationale for government intervention vanishes. In these limiting cases, the optimal policy is* laissez-faire, *characterized by a zero tax rate ($x = 0$) and the elimination of the student grant ($g = 0$).*

Proposition 3 highlights the contingency of optimal education policy on the underlying growth engine of the economy. The vanishing rationale for intervention in these two limiting cases offers several key insights:

The growth-dependency of policy. In a stationary economy where technological progress is absent, the intertemporal benefits of schooling are confined to private productivity gains. Without the "innovation lever", there are no dynamic social spillovers for the government to internalize. Consequently, the social rate of return on education aligns perfectly with the private return, and any fiscal intervention – through subsidies or taxes – would only introduce deadweight loss by distorting the natural allocation of labor.

The structural link to R&D. Even in a rapidly growing economy, the justification for intervention depends on the type of growth. If R&D productivity is independent of schooling duration – implying that education does not enhance the innovative capacity of skilled workers – then the "intensive margin" of education generates no positive externalities for the R&D sector. In this scenario, education behaves as a standard private good. The government's role as a social coordinator becomes redundant because the individual's choice of schooling duration does not influence the economy's aggregate technological frontier.

The neutrality of laissez-faire. These limiting cases serve as a "neutrality benchmark". They demonstrate that the optimal policy package derived in this study is not a general result of individual heterogeneity or monopolistic competition alone. Instead, it is a specific corrective response to the growth-induced distortion created by the link between human capital and innovation.

In summary, the transition from a passive *laissez-faire* regime to an active policy framework is driven entirely by the synergy between educational duration and the R&D sector's efficiency. As the economy moves away from these limits, the "policy wedge" must be introduced to bridge the gap between private incentives and the social needs of a knowledge-driven economy.

Result (39) can be rephrased as follows:

Proposition 4 *The socially optimal student grant* (39) *is an increasing function of the R&D efficiency* δ*, R&D input* v*, and the elasticity of R&D efficiency with respect to education duration,* ε_D*. Conversely, it is a decreasing function of the interest rate* r *and the mass of students,* $(m-b)D$*.*

Proposition 4 highlights the following tension between the social returns and the social costs of human capital accumulation.

Factors that enhance the productivity of knowledge – specifically R&D efficiency δ, higher R&D investment v, or a greater elasticity ε_D – raise the social value of a skilled worker. Because individuals do not internalize how their education benefits the broader R&D sector (a positive externality), the social planner must bridge this gap. A higher grant g serves as the necessary subsidy to align private incentives with the social optimum.

In growth models, a higher interest rate r increases the relevant discount rate. This implies that the future benefits of education – such as higher wages and accelerated growth – are worth less in present value terms. Effectively, a higher r increases the opportunity cost of education relative to immediate production. Furthermore, if the mass of students, $(m-b)D$, is already substantial, the marginal cost of further education rises due to resource constraints or diminishing returns in the labor mix. In such scenarios, the planner reduces the grant g to prevent inefficiently high levels of investment.

Result (40) can be rephrased as follows:

Proposition 5 *The socially optimal tax rate on skilled labor income,* (40), *increases with the aggregate amount of the student's grants, wgD, and the effective discount rate $r + m$.*

This result reflects both fiscal and temporal trade-offs. First, because the model lacks non-distorting (lump-sum) taxes, then – to satisfy the government's budget constraint – any increase in total grant expenditures wgD necessitates higher tax rates on both skilled labor income x and consumption φ. Second, a higher effective discount rate $r+m$ lowers the present value of future labor income, which reduces the social efficiency of prolonged education. In this context, the planner raises the income tax on skilled labor, x, as a corrective instrument to discourage excessively long educational tenures and facilitate a more timely transition into the productive workforce.

7 Conclusions

This study characterizes the optimal policy in an endogenous growth model where heterogeneous individuals choose between entering the workforce as unskilled labor or pursuing education to become skilled. Within a stationary population, we model a decentralized economy where monopolists produce goods using both labor types and purchase innovations from an R&D sector using only skilled labor. Crucially, R&D productivity depends on education duration under the supervision of senior experts.

We identify two fundamental market distortions: the *extensive margin* (the decision to acquire skills) and the *intensive margin* (the duration of education). By deriving the social planner's benchmarks, we demonstrate that optimal education and labor allocation are achieved when marginal social benefits – discounted by interest and mortality rates – align with marginal social costs, including the positive externalities generated by R&D efficiency.

In a decentralized equilibrium, the two distortions are corrected through two distinct instruments: a student grant and a targeted income tax on skilled labor. However, because individuals are heterogeneous and the optimal grant cannot cover all education costs, emerging liquidity constraints must be relaxed via government-backed student loans. The rationale for these interventions is contingent upon the presence of economic growth and the functional link between education and innovation. We demonstrate that in two limiting cases – where economic growth is absent or where R&D productivity is independent of education duration – the justification for government intervention vanishes. In these scenarios, the externalities generated by the schooling process are nullified, making a laissez-faire approach – defined by a zero tax rate on skilled labor and the removal of student grants – socially optimal. Consequently, the optimal policy package is highly sensitive to the economy's innovation intensity.

The results of this study underscore the dual role of fiscal policy in managing human capital in a growth-oriented economy. The student grant serves as a primary tool for internalizing the positive externalities of education. When the social rate of return on schooling is high – driven by rapid growth, intensive R&D investment, or high R&D elasticity – the optimal grant increases to bridge the gap between private incentives and social needs. Conversely, when the opportunity costs of education rise due to high interest rates or a saturated student population, the planner reduces the grant to prevent the misallocation of labor.

The income tax on skilled labor serves as a critical corrective instrument for the timing of labor market entry. Because of endogenous technological change and individual heterogeneity, non-distorting taxes cannot be utilized, and the planner must balance the budget using both skilled labor and consumption taxes. Crucially, the skilled labor tax is primarily employed to moderate the intensive margin: as the effective discount rate rises, the social demand for immediate production increases. By raising the tax on skilled labor income, the planner reduces the marginal benefit of prolonged schooling, thereby mitigating over-education and facilitating a timely transition into the productive workforce.

Ultimately, this paper suggests that education policy cannot be designed in isolation from the macroeconomic environment. The optimal policy package is sensitive to the underlying drivers of growth: as an economy shifts toward higher innovation intensity, the mechanism for managing both the number of students and their time in the classroom must adapt accordingly.

Appendix

A The investor's equilibrium (eq. 21)

Monopolist i maximizes its present value (20) subject to (18) by z_i, given (r, w, p, y, a, P_i, D). The Hamiltonian corresponding to this maximization is

$$\mathcal{H} \doteq \Pi(a_i, w, p, y) - P_i z_i + \beta \dot{a}_i = \Pi(a_i, w, p, y) - P_i z_i + \beta z_i a, \tag{41}$$

where the co-state variable β evolves according to

$$\dot{\beta} = r\beta - \frac{\partial \mathcal{H}}{\partial a_i} = r\beta - \frac{\partial \Pi}{\partial a_i}, \quad \lim_{t\to\infty} \beta a_i e^{-rt} = 0. \tag{42}$$

The first-order condition of maximizing the Hamiltonian (41) is

$$0 = \frac{\partial \mathcal{H}}{\partial z_i} = \beta a - P_i \;\;\Leftrightarrow\;\; \beta = \frac{P_i}{a}. \tag{43}$$

This result can be interpreted as follows. As long as $\beta > \frac{P_i}{a}$ and $\frac{\partial \mathcal{H}}{\partial z_i} > 0$ hold true, investors increase their demand for effective skilled labor to produce new patents. Conversely, if $\beta < \frac{P_i}{a}$ and $\frac{\partial \mathcal{H}}{\partial z_i} < 0$ hold true, investors cease employing effective skilled labor, setting $z_i = 0$. Consequently, the market rapidly adjusts to the equilibrium $\beta = \frac{P_i}{a}$, where $\frac{\partial \mathcal{H}}{\partial z_i} = 0$.

In the steady state of the economy, the interest rate r, the relative wage w and the quantity of patents, z_i, must be constants. This means by (42) that the product of the co-state variable β and the state-of-the-art efficiency a, βa, must be constant. Therefore, if a grows at the constant rate γ in the economy, then β grows at the rate $-\gamma$ and from (42) it follows that

$$\gamma = \frac{\dot{a}}{a} = -\frac{\dot{\beta}}{\beta} = \frac{1}{\beta}\frac{\partial \Pi}{\partial a_i} - r = \frac{a}{P_i}\frac{\partial \Pi}{\partial a_i} - r \;\;\Leftrightarrow\;\; P_i = \frac{a}{\gamma + r}\frac{\partial \Pi}{\partial a_i}.$$

B The cohort's equilibrium (eqs 32-34)

B.1 Optimization

The cohort born at time $t = 0$ maximizes its utility (30) by selecting the share of its members who remain workers, b, and the education duration D from the *set of constant controls*, and the consumption path $c(\xi)$ for $\xi \in [0, \infty)$. The maximization is subject to constraints (2), (4), and (31), given the parameters $(\Theta, r, p, w, g, \varphi)$.[9] Because the constraints (4) and (31) are *integro-differential equations,* then, according to Hartl and Sethi (1984) [or Grass et al. (2008), p. 423] and Bokov (2011), the Hamiltonian of the problem of the cohort born at time $t = 0$ can be specified as follows:[10]

$$\begin{aligned}
\Omega &\doteq \ln c + \mu_l(b - ml) + \mu_s[h(b, D) - ms] + \\
&\mu_W\left\{rW + \Theta + l + (1 - x)\left[ws + (g - \alpha)w\int_0^D (m - b)d\xi\right] - (1 + \varphi)pc\right\} \\
&= \ln c + \mu_l(b - ml) + \mu_s[h(b, D) - ms] + \\
&\quad \mu_W\big\{rW + \Theta + l + (1 - x[ws + (g - \alpha)(m - b)wD] - (1 + \varphi)pc\big\}, \quad (44)
\end{aligned}$$

where the co-state variables (μ_W, μ_l, μ_s) evolve as follows:

$$\dot{\mu}_l = \rho\mu_l - \frac{\partial\Omega}{\partial l} = (r + m)\mu_l - \mu_W, \tag{45}$$

$$\dot{\mu}_s = \rho\mu_s - \frac{\partial\Omega}{\partial s} = (r + m)\mu_s - (1 - x)wv_W, \tag{46}$$

$$\dot{\mu}_W = \rho\mu_W - \frac{\partial\Omega}{\partial W} = (\rho - r)\mu_W. \tag{47}$$

The transversality conditions are given by $\lim_{t\to\infty} \mu_l(t)l(t)e^{-\rho t} = 0$, $\lim_{t\to\infty} \mu_s(t)s(t)e^{-\rho t} = 0$ and $\lim_{t\to\infty} \mu_W(t)W(t)e^{-\rho t} = 0$. The first-order

[9]The notation in the model is simplified by omitting the current time t from the functions of time – i.e., $W(t) = W$, $l(t) = l$, $s(t) = s$, $c(t) = c$, $b(t) = b$, $\mu_W(t) = \mu_W$, $\mu_l(t) = \mu_l$ and $\mu_s(t) = \mu_s$ – for convenience.

[10]Hartl and Sethi (1984) established this result for optimal control problems with a finite time horizon. Under the assumptions that the state variables (l, s, W) are bounded over time, and that the function $h(b, D)$ and with its partial derivatives are bounded, Bokov subsequently extended this result to infinite-horizon problems. These conditions are satisfied in the present model.

conditions for maximizing the Hamiltonian (44) are

$$
\begin{aligned}
0 &= \frac{\partial \Omega}{\partial D} = h_D(b, D)\mu_s + (1-x)(g-\alpha)(m-b)w\mu_W \\
\Leftrightarrow \quad & h_D = -(1-x)(g-\alpha)(m-b)w\mu_W/\mu_s, && (48)\\
0 &= \frac{\partial \Omega}{\partial b} = h_b(b, D)\mu_s - (1-x)(g-\alpha)wD\mu_W + \mu_l \\
\Leftrightarrow \quad & h_b = (1-x)(g-\alpha)wD\mu_W/\mu_s - \mu_l/\mu_s, && (49)\\
0 &= \frac{\partial \Omega}{\partial c} = \frac{1}{c} - (1+\varphi)p\mu_W \quad \Leftrightarrow \quad \mu_W = \frac{1}{(1+\varphi)pc} > 0. && (50)
\end{aligned}
$$

B.2 Steady state

In the steady state, labor inputs (l, n) are constants. Then, according to (23), (26) and (27), the product pc is a constant, which means that the price p decreases at the rate $\frac{\dot{p}}{p} = -\frac{\dot{c}}{c} = -\frac{\dot{a}}{a} = -\gamma$. Then, from (47) and (50) it follows that μ_W is a constant. Combining this with (45)-(47) yields

$$
\begin{aligned}
& \dot{\mu}_W = 0 \quad \Leftrightarrow \quad r = \rho, && (51)\\
& 0 = \frac{\dot{\mu}_W}{\mu_W} = \frac{\dot{\mu}_l}{\mu_l} = r + m - \frac{\mu_W}{\mu_l} \quad \Leftrightarrow \quad \frac{\mu_W}{\mu_l} = r + m, && (52)\\
& 0 = \frac{\dot{\mu}_W}{\mu_W} = \frac{\dot{\mu}_s}{\mu_s} = r + m - (1-x)w\frac{\mu_W}{\mu_s} \quad \Leftrightarrow \\
& \frac{\mu_W}{\mu_s} = \frac{r+m}{(1-x)w} = \frac{1}{(1-x)w}\frac{\mu_W}{\mu_l} \quad \Leftrightarrow \quad \frac{\mu_l}{\mu_s} = \frac{1}{(1-x)w}. && (53)
\end{aligned}
$$

Inserting (52) and (53) into the first-order conditions (48) ans (49) yields

$$
h_D = -(g-\alpha)(m-b)\underbrace{(1-x)w\frac{\mu_W}{\mu_s}}_{=r+m} = (\alpha - g)(m-b)(r+m), \qquad (54)
$$

$$
\begin{aligned}
& h_b = (g-\alpha)D\underbrace{w(1-x)\frac{\mu_W}{\mu_s}}_{r+m} - \underbrace{\frac{\mu_l}{\mu_s}}_{=\frac{1}{(1-x)w}} = (g-\alpha)D(r+m) - \frac{1}{(1-x)w} \\
\Leftrightarrow \quad & \frac{1}{r+m}\frac{1}{(1-x)w} = (g-\alpha)D - \frac{h_b}{r+m} = (g-\alpha)D + \frac{|h_b|}{r+m}. && (55)
\end{aligned}
$$

Equations (51), (54) and (55) yield (32)-(34).

C The social planner's equilibrium (eqs 36)

Because (b, D) are selected from the set of constant controls, constraints (2), (4), (23) and (28) take the form

$$\dot{l} = b - ml, \quad \dot{s} = h(b, D) - ms, \quad \dot{a}/a = \delta(D)v,$$
$$s - n - v = \alpha \int_t^D [m - b(\xi)]d\xi = \alpha D(m - b), \quad z \geq 0. \tag{56}$$

The social planner maximizes utility (30) subject to (56). Because one constraint is an *integro-differential equation,* according to Hartl and Sethi (1984) [or Grass et al. (2008), p. 423] and Bokov (2011), the Hamiltonian of this maximization can be specified as follows:[11]

$$\Lambda \doteq U(a, l) + \lambda_l(b - ml) + \lambda_s[h(b, D) - ms] + \lambda_a\delta(D)va$$
$$= \ln a + \ln f(l, n) + \lambda_l(b - ml) + \lambda_s[h(b, D) - ms(t)] + \lambda_a a\delta(D)v, \tag{57}$$

where $(\lambda_l, \lambda_s, \lambda_a)$ are the co-state variables for the state variables (l, s, a). The planner maximizes the Hamiltonian (57) subject to constraints (56). Because in (56) there is one equality constraint and one inequality constraint, the Lagrangean of this maximization is the following:

$$\mathcal{L} \doteq \Lambda + \chi[s - n - v - \alpha D(m - b)] + \zeta v. \tag{58}$$

which is maximized by (n, b, t, v) subject to the Kuhn-Tucker conditions

$$\chi[s - n - v - \alpha D(m - b)] = 0, \quad \zeta z = 0, \quad \zeta \geq 0, \tag{59}$$

where χ and ζ are the Kuhn-Tucker multipliers.

The first-order conditions for maximizing the Lagrangean (57) are

$$0 = \frac{\partial \mathcal{L}}{\partial n(t)} = \frac{\partial \Lambda}{\partial n(t)} - \chi = \frac{f_n}{f} - \chi \quad \Leftrightarrow \quad \chi = \frac{f_n}{f}, \tag{60}$$
$$0 = \frac{\partial \mathcal{L}}{\partial b(t)} = \frac{\partial \Lambda}{\partial b(t)} + \alpha D\chi = \lambda_s h_b + \lambda_l + \alpha D\frac{f_n}{f} \quad \Leftrightarrow \quad h_b = -\frac{\alpha D}{\lambda_s}\frac{f_n}{f} - \frac{\lambda_l}{\lambda_s}, \tag{61}$$

[11]The notation is simplified by omitting the current time t from the functions of time – i.e., $a(t) = a$, $l(t) = l$, $s(t) = s$, $l(t) = l$, $c(t) = c$, $b(t) = b$, $\lambda_s(t) = \lambda_s$, $\lambda_l(t) = \lambda_l$ and $\lambda_a(t) = \lambda_a$ – for convenience.

$$0 = \frac{\partial \mathcal{L}}{\partial D(t)} = \frac{\partial \Lambda}{\partial D(t)} - \alpha(m-b)\chi = \lambda_s h_D + \lambda_a a\delta' v - \alpha(m-b)\frac{f_n}{f}$$

$$\Leftrightarrow \ h_D = \frac{\alpha}{\lambda_s}\frac{f_n}{f}(m-b) - \frac{\lambda_a}{\lambda_s}a\delta' v, \tag{62}$$

$$0 = \frac{\partial \mathcal{L}}{\partial v(t)} = \frac{\partial \Lambda}{\partial v(t)} + \zeta - \chi = \lambda_a a\delta + \zeta - \chi \ \Leftrightarrow \ \zeta = \chi - \lambda_a a\delta. \tag{63}$$

The co-state variables $(\lambda_l, \lambda_s, \lambda_a)$ evolve according to [cf., (56), (60)]

$$\dot{\lambda}_l = r\lambda_l - \frac{\partial \Lambda}{\partial l} = (r+m)\lambda_l - \frac{f_l}{f}, \tag{64}$$

$$\dot{\lambda}_s = r\lambda_s - \frac{\partial \Lambda}{\partial s} = (r+m)\lambda_s - \chi = (r+m)\lambda_s - \frac{f_n}{f}, \tag{65}$$

$$\dot{\lambda}_a = r\lambda_a - \frac{\partial \Lambda}{\partial a} = (r-\delta z)\lambda_a - \frac{1}{a} = (r-\delta(D)v)\lambda_a - \frac{1}{a}. \tag{66}$$

The transversality conditions of the problem are $\lim_{t\to\infty}\lambda_l(t)l(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_s(t)s(t)e^{-\rho t} = 0$ and $\lim_{t\to\infty}\lambda_a(t)a(t)e^{-\rho t} = 0$.

In the steady state of the system (2), (4), (23), (26) and (59)-(66), the control variables (n, b, D), state variables (l, s), the co-state variables (λ_l, λ_s) and the product $\lambda_a a$ are all constants, while consumption c and productivity a grow at the same constant rate $\dot{c}/c = \dot{a}/a = \delta(D)v$. This yields the steady-state conditions

$$r - \delta(D)v - \frac{1}{\lambda_a a} = \frac{\dot{\lambda}_a}{\lambda_a} = -\frac{\dot{a}}{a} = -\frac{\dot{c}}{c} = -\delta(D)v \ \Leftrightarrow \ \lambda_a a = \frac{1}{r},$$

$$\dot{l} = 0 \ \Leftrightarrow \ b = ml, \quad \lambda_l = 0 \ \Leftrightarrow \ \lambda_l = \frac{f_l/f}{r+m}, \quad \lambda_s = 0 \ \Leftrightarrow \ \lambda_s = \frac{f_n/f}{r+m},$$

$$\dot{s} = 0 \ \Leftrightarrow \ h = ms. \tag{67}$$

Inserting (67) into the first-order conditions (61) and (62) yield

$$|h_b| = -h_b = \frac{\alpha D}{\lambda_s}\frac{f_n}{f} + \frac{\lambda_l}{\lambda_s} = (r+m)\alpha D + \frac{f_l}{f_n} \ \Leftrightarrow \ \frac{|h_b|}{r+m} = \alpha D + \frac{f_l/f_n}{r+m},$$

$$h_D = \frac{\alpha}{\lambda_s}\frac{f_n}{f}(m-b) - \frac{\lambda_a}{\lambda_s}a\delta' v = (r+m)(m-b)\alpha - (r+m)\frac{f}{f_n}\frac{1}{r}\delta' v \ \Leftrightarrow$$

$$\frac{h_D}{r+m} = (m-b)\alpha - \frac{f}{f_n}\frac{1}{r}\delta' v.$$

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