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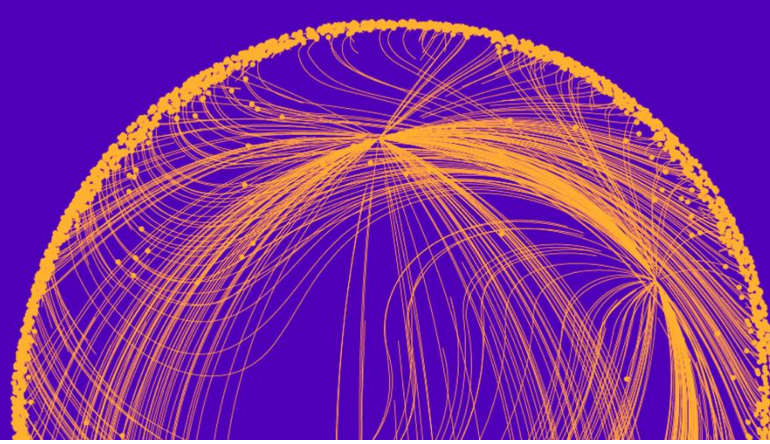
Heterogeneous price commitments

Saara Hämäläinen



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Heterogeneous price commitments

Saara Hämäläinen*†

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Abstract

We provide a theory of dynamic oligopoly pricing with heterogeneous price technologies, captured by the presence of *trackers* (on the firm-side) and *shoppers* (on the consumer-side) who can costlessly follow market prices. Our paper contributes to reconciling the variation in observed price dynamics. Due to the limit pricing resembling price behavior of non-trackers, the equilibrium price distribution can feature gaps and an atom. A price war may erupt, collusion is possible, or prices can remain unchanged for a while. Since the equilibrium price distribution is contingent on the number of trackers, our study puts forth several testable hypotheses to explore the impact of trackers. We also find that, although tracking diminishes consumer welfare and incentives for searching, trackers only benefit from the wider presence of other trackers up to a point.

Keywords: *Dynamic oligopoly, price dispersion, price monitoring, commitment, price cycles, collusion, limit pricing.* **JEL-codes:** D43, D83, L11, L41.

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1 Introduction

Homogeneous goods and close substitutes are frequently available at persistently dissimilar prices across firms (Baye et al., 2006a,b; Chandra and Tappata, 2011; Kaplan and Menzio, 2015; Gorodnichenko and Talavera, 2017). Classic sales models, following Varian (1980), provide a partial explanation of equilibrium price dispersion based on a firm's uncertainty about consumer price information (Bergemann et al., 2021) or commitment to search prices (Stahl, 1989). By randomizing between high (monopoly) prices and lower (discount) prices, firms can alternate in targeting both uninformed and informed consumers, without knowing which consumers they face. Nevertheless, the firm-side uncertainty about competitor price information or its commitment to follow market prices has been relatively overlooked. If a rival can instantaneously match any price, but the firm itself must commit to its price for a while, only monopoly pricing to uninformed consumers or Bertrand pricing over informed consumers can be sustained. Consequently, price dispersion wanes.

Firms are known to track market prices fairly intensively, basically, by accessing the same information channels as consumers, e.g., shopping apps or comparison sites. Continuous monitoring has the advantage of enabling a firm to respond immediately to any change in market prices.[1] Nonetheless, data show that rival firms may also opt to keep their prices fixed for a prolonged time (Figure 1).[2] The goal of this paper is to understand how uncertainty about competitors' price information and price commitment affects the prices of homogeneous products in search markets where consumers are generally heterogeneously informed about prices.

To achieve this goal, our paper develops a framework where both consumers and firms make price observations. A fraction of consumers (i.e., *shoppers*) and a share of firms (i.e., *trackers*) can view all prices in the market at zero cost. Non-shoppers (e.g., consumers with positive search costs) observe one price quote. Non-trackers (e.g., firms with positive menu costs) only observe their own price. In a dynamic pricing game, non-trackers commit to prices at $t = 0$ and keep them fixed until $t = 1$; trackers can change their prices at any $t = \frac{1}{n}, ..., \frac{n-1}{n}$. A consumer may arrive in the market at any time, when her need for purchasing a product emerges.[3]

[1]There exist various tools that enable firms to track rival prices, receive alerts of price drops, allow automated repricing, etc. (`https://visualping.io/blog/top-tools-competitor-price-tracking/`, accessed Nov 29, 2022).

[2]The sector inquiry of the European Commission (EC) found that 53 % of retailers tracked competing prices; 67 % of trackers used automated software to follow market prices while 78 % of software users adjusted their prices to rival prices. A possible scenario expressed in the memo is that small firms act as price leaders while big e-commerce firms commit to follow. (`https://ec.europa.eu/competition/cartels/icn/szekely.pdf`, accessed Feb 24, 2023.)

[3]Model time frame reflects the time for which non-trackers' prices remain fixed. This model time window could equally well be a minute or a month. The model can be repeated to allow also non-trackers to change their prices later.

We find that price behavior changes quite remarkably when we allow some firms to observe changes in market prices almost in real-time. Only non-trackers price in mixed strategies. Trackers' prices inherit the appearance or randomness from the non-trackers' prices they follow but can remain constant for extended periods as long as non-trackers keep their prices fixed.[4] As a novelty, the equilibrium price distribution may also display *gaps* at certain intermediate prices and has an *atom* at the monopoly price.

Because equilibrium pricing is shaped by non-trackers' incentives to restrict their exposure to trackers' collusion, the most interesting price strategies require at least three firms whose types remain uncertain initially. The prospect of competing with another non-tracker entails, by standard reasoning, that a non-tracker optimally prices in randomized strategies. By contrast, trackers can easily learn rival types and observe rival prices, which allows a tracker to undercut non-trackers and collude with trackers. However, the incentives for undercutting are obviously lower if there are many trackers sharing collusive profits.[5] As we demonstrate in this work, non-trackers may thus choose to set such low prices that only a tracker who does not face competition from rival trackers would be willing to price below them. Instead, multiple trackers would rather choose the monopoly price and concentrate on targeting their captives. This equilibrium market segmentation gives rise to the price atom. Another novelty we find is that, because higher prices expose non-trackers to significantly stronger competition from colluding trackers, there is a gap in the equilibrium price distribution between the higher prices that at most r rival trackers would undercut and the lower prices that at most $r-1$ rival trackers would undercut, etc.[6] Moreover, when competition from trackers is harshest, non-trackers may only employ the lower prices that colluding trackers are not willing to undercut, resulting in a pattern akin to limit pricing.[7] This does not arise in static models of sales.

These new model features prove helpful for reconciling certain salient features of observed price dynamics in a simple unified framework. To illustrate the puzzling variation of pricing patterns encountered in oligopolistic online markets, Figure 1 depicts the price movements of books available at Amazon compared to the best third-party prices.[8] Authentic price patterns are generally much richer than the ones predicted by the repeated application of clearinghouse search models, such as Burdett and Judd (1983).

1. *Fixed price.* Prices fluctuate between a high (regular) price and a low (discount) price but habitually remain unchanged for a while at each price level (Figs. 1a).

2. *Atom(s) and gap(s).* Some price have, on the one hand, a tendency to revert to a

[4]In general, as expected, equilibrium price dispersion arises without future punishments if there are also non-shoppers and non-trackers.

[5]Or worse, rival trackers willing to initiate a price war cycle.

[6]In a triopoly, $r = 2$ but the idea generalizes with more firms.

[7]The mechanism is however different from the standard one (Milgrom and Roberts, 1982) because we do not study market entry deterrence.

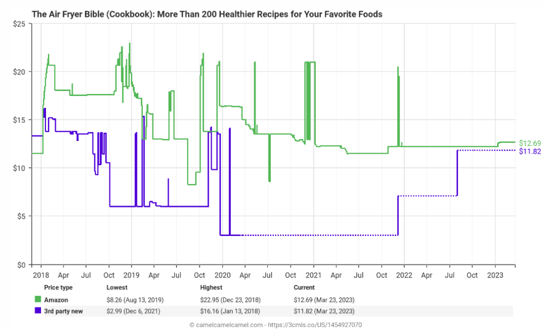
[8]Source: *Camelcamelcamel* (`https://camelcamelcamel.com`, accessed: Jan 30 and Mar 23, 2023).

particular level repeatedly after a brief change while, on the other hand, persistently shunning certain intermediate levels (Figs. 1a-d).

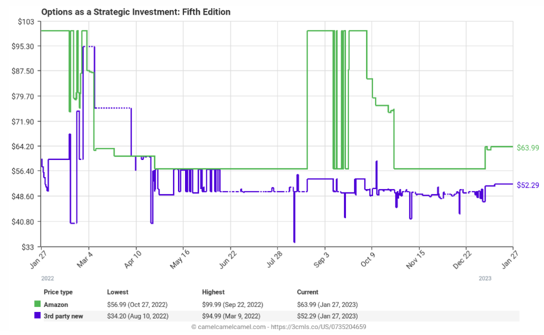
3. *Pricing below rival.* Prices appear to be randomized either a lot below or a bit below a higher-priced rival who does not respond by cutting its own price (Fig. 1b).

4. *Cyclic price wars and possible tacit collusion.* Rival firms keep alternately undercutting each other's prices. Thereafter, either a price increases and the cycle restarts or prices remain fixed close to each other for a while (Fig. 1c).

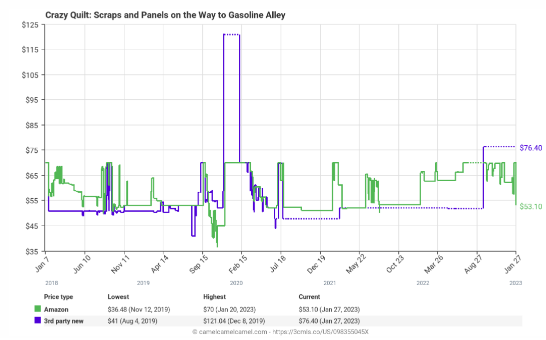
5. *Classic mixed prices.* Rival firms' prices fluctuate apparently independently over the same interval (Fig. 1d).



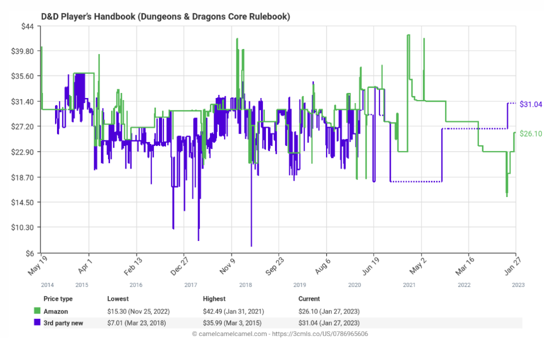
(a) Fixed price, price atoms, and gap



(b) Mixing below rival



(c) Price war cycle and market sharing



(d) Mixing on interval

Figure 1: Third party (blue) and Amazon prices (green) at *Camelcamelcamel.com*.

The final pattern comes closest to the predictions of the basic Varian (1980) model whereas the price war cycles resemble classic Edgeworth cycles (Maskin and Tirole, 1988). Price rigidity is generally regarded as a natural feature of tacit collusion (Green and Porter, 1984; Athey et al., 2004; Jullien and Rey, 2007). However, a distinctive characteristic of this price data is that a firm may keep its price fixed at an intermediate level while its rivals alter theirs (e.g., Fig. 1c). This is consistent with our model of heterogeneous price commitments but otherwise hardly explained.[9] Moreover, we see that a firm who

[9]Of course, many price patterns still remain unexplained by our purposely stylized model, which fails to account, e.g., learning over product life-cycle (Bergemann and Välimäki, 2006), information costs of price adjustment Alvarez et al. (2011), and strategic timing of purchase choices (Garrett, 2016).

is a likely tracker (Amazon), underprices its competitors when its rivals have high prices but keeps its price fixed at a higher level when rival prices remain lower (e.g., Fig. 1b). This pattern with a gap and an atom is predicted by our model of heterogeneous price commitments. By contrast, gaps fail to arise in most clearinghouse models of search and atoms only arise in models of asymmetric firm prominence (Narasimhan, 1988; Wilson, 2010; Hämäläinen, 2018).[10] A straightforward model extension shows that heterogeneous price commitments arise in our framework if firms have dispersed menu costs.

Regulatory implications

Pricing and tracking can either be implemented by people or delegated to algorithms. Price monitoring and price algorithms are currently under scrutiny for their potentially significant role in facilitating collusion in electronic marketplaces.[11] We contribute to the discussion by offering a theory of pricing with heterogeneous price technologies and showing how differences among firms impinge on existing theories of dynamic oligopoly pricing, the telltale signals of competitive vs. collusive pricing, and the sustainability of collusive behavior.

Our paper describes a mechanism based on firm heterogeneity that naturally limits collusive prices in a mixed oligopoly where both trackers and non-trackers are present at the same time. In general, we observe that a tracker either (i) charges the monopoly price targeting its captive non-shoppers; (ii) undercuts the lowest current rival price to attract all shoppers; or (iii) colludes with trackers to a price that slightly undercuts the lowest price among non-trackers. As a conclusion, because the lowest price among non-trackers defines an upper bound on the prices that appeal to shoppers, we find that collusive prices lie strictly below the monopoly price in a market with trackers and non-trackers. Under heterogeneous price commitments, a notable feature of pricing is, therefore, that collusive prices remain *low*. This restricts the profitability of collusion.

The presence of non-trackers can also render tracker collusion harder to sustain because non-trackers may commit to low prices to restrict their exposure to competition, paving the way for multiple equilibria, with varying degrees of collusiveness. The least collusive of these equilibria resemble limit pricing, garnering a higher profit for non-trackers against multiple trackers. Non-trackers then charge such inexpensive prices that only a single tracker is willing to meet them. If the price is low enough, multiple trackers are not interested in colluding at any price below it because the demand gain from shoppers would be split between the trackers. Rival trackers are therefore better off if they each

[10]But see Hämäläinen (2022) for atomistic pricing in symmetric equilibria.

[11]The European Commission (EC) identifies price monitoring in retail price maintenance (RPM) as a special concern. For example, the EC imposed a total of 111 million EUR in fines on consumer electronics companies in 2018, for using sophisticated price monitoring algorithms to follow retail prices, allowing them to intervene fast in cases where retailers decreased prices.

concentrate on selling to their captives at the monopoly price. Unlike collusive pricing, this concerted monopoly pricing coincides with *low* tracker profit.

Our results stretch the conventional idea that optimal collusive prices must be high and simultaneous monopoly pricing by rival firms indicative of collusion. This has practical relevance for, e.g., the auditing of pricing algorithms for collusion (Calvano et al., 2020) and updating collusive pricing detection rules (Harrington, 2018). Overall, our results show that consumers are generally hurt by the presence of firms whose price commitments have a shorter duration. However, because the negative price effects of tracking are broader than collusion-related, regulators cannot tackle the problems solely by looking for signals of collusion.

Regulating the frequency of price change offers one alternative that should perhaps be tested carefully in the lab. For empirical research, our research suggests an immediate way to test how strongly the market prices are affected by tracking:[12] In a market with few trackers, the equilibrium generates a smooth price histogram with a tiny gap at middle prices and a tiny mass at the highest price. Consumer welfare in then relatively high. When trackers is numerous, our testable hypothesis is that the distribution of prices is almost binary with a large gap between the lowest and highest prices. This case suggests reduced consumer surplus.

Literature contribution

This work contributes to the theory of dynamic oligopoly pricing and algorithmic pricing under consumer search. To clarify our contribution to the literature and highlight the main new features, we next contrast our results with the established theories of oligopolistic pricing.

Commitment in dynamic oligopoly pricing. Maskin and Tirole (1988) provide a general theory of dynamic oligopoly with an alternating offer setup, where all price commitments are short-lived. Edgeworth cycle and collusive pricing arise as Markov-perfect equilibria. In this paper, we extend the analysis by allowing for prolonged commitments and consumer heterogeneity, which profoundly alters equilibria. The value of commitment and costs of non-commitment are pointed out perhaps earliest by Stackelberg (1934) in his criticism of Cournot competition and by Coase (1972) in his treatment of dynamic monopoly. Our model features imperfectly observed price commitments considered more thoroughly in Van Damme and Hurkens (1997). Van Damme and Hurkens (1996) prove that mixed equilibria are commitment robust when no firm has a first-mover advantage.[13] Here, heterogeneity on each market side sustains mixed strategies, although non-trackers can benefit from moving before trackers. In the classic static models of Butters (1977);

[12] For an example and review of how the testing might proceed, see Berck et al. (2008).

[13] Endogenous price leadership is considered in Van Damme and Hurkens (2004) and Deneckere and Kovenock (1992). In our model, non-trackers are always price leaders.

Varian (1980); Burdett and Judd (1983); Stahl (1989) price dispersion arises due to heterogeneous consumer price information. The persistence of equilibrium price dispersion in a dynamic setting where firms become aware of rival prices remains an open research question.

Dynamic price dispersion with consumer search. Myatt and Ronayne (2019) offer a theory of stable equilibrium price dispersion, based on commitments to list prices that can only be discounted but not increased. Unlike here, only two pure pricing strategies prevail in equilibrium: a leader firm sets a discount price whereas the followers set a regular price. In our model, equilibrium price dispersion is driven by non-trackers' mixed strategies but persists long-term because of trackers' pure strategies. The general problem with stable equilibrium price dispersion is spelled out in Lahkar (2011). That is, evolution favors cycles. We sidestep this problem by endowing some firms with the quite realistic power to commit to prices for a moment. Heidhues and Kőszegi (2008) demonstrate that consumer loss aversion can eliminate price dispersion and lead to focal (uniform) price equilibria.

Dynamic algorithmic pricing. The closest article to ours in the literature is perhaps Brown and MacKay (2023) who study competitive pricing by pricing algorithms that either have different pricing frequencies or allow for automatic updating in response to the rival price. Prices are proved to lie between Bertrand and Stackelberg prices, which shows that algorithms can increase prices without collusion; collusion is possible as well. The paper abstracts from consumer heterogeneity and introduces firm heterogeneity very differently. The remaining literature is focused on symmetric price-setting technologies among firms. Calvano et al. (2020, 2021) observe that pricing algorithms quickly learn to charge collusive prices sustained by brief punishment phases. Assad et al. (2020) observe that duopoly prices increase only if both firms adopt algorithmic pricing techniques. Cason and Friedman (2003) and Cason et al. (2021) provide laboratory evidence of price dispersion, price correlation and cyclic prices in the classic Burdett and Judd (1983) noisy search model. Zhang and Feng (2011) identify cyclical bid adjustments in search engine advertising data.

Collusive price strategies with consumer search. Petrikaitė (2016) demonstrates for homogenous products that cartels become less stable as search costs increase. Schultz (2017) observes that profits from collusion increase in consumer information but decrease in firm information.[14] Montag and Winter (2020) consider the welfare effects of mandatory price disclosure. The main difference in our paper is that we study markets with equilibrium price dispersion, where collusive prices generally remain below other market prices limiting collusive profits. A few recent papers study collusion with price dispersion. In de Roos and Smirnov (2020) intertemporal price dispersion facilitates collusion

[14] In a classic paper, Green and Porter (1984) show that less transparent demand reduces collusive profits by increasing the prevalence of punishments. In follow-up work, Garrod and Olczak (2021) find that more transparent supply boosts collusive profits by restricting the duration and frequency of punishments.

by obfuscating the price comparisons made by consumers. In Shadarevian (2022) welfare considerations result in dispersed collusive prices with patient firms and variable costs.

The article is organized as follows. The basic (duopoly) model is spelled out in Section 2. Section 3 describes duopoly equilibria and Section 4 considers triopoly equilibria. Section 5 concludes by discussing some extensions and alternative model assumptions. Proofs are contained in the Appendix.

2 Model

Our model can be regarded as a marriage between the Varian (1980) model of heterogeneous price information and the Maskin and Tirole (1988) model of alternating price proposals.

We initially study oligopoly price competition in a market with two firms $i = 1, 2$ and a consumer. Each firm is selling one product, of which the consumer intends to buy one. The surplus of trading a product is fixed at unity. A firm's profit share is given by its price $\pi^i = p^i \in [0, 1]$ and the consumer utility denoted by $u(p^i) = 1 - p^i \in [0, 1]$.[15]

Shoppers and non-shoppers. Firms are uncertain regarding (i) the consumer's price information and (ii) the rival's price technology. The consumer has two possible price information types. A shopper observes both firms' prices whereas a non-shopper only observes one of the prices in the market at random. The probability that the consumer is a shopper is $\mu > 0$.

Trackers and non-trackers. Firms have two feasible price technology types, which affect their ability to commit to prices. A tracker observes the rival's price and can reset its price following the price observation. A non-tracker either cannot observe the rival's price or, equivalently, cannot reset its own price. The likelihood that a firm is a tracker is denoted by $\tau > 0$.

Heterogeneous price commitments. Our analysis is focused on studying firm behavior over the time interval $[0, 1]$, whose length corresponds with the time for which a non-tracker's price remains fixed. To simplify, we assume that all non-trackers choose their prices simultaneously at the beginning of the interval. By contrast, trackers can change their prices multiple times during this interval while non-tracker prices remain unchanged.[16] The consumer may arrive in the market at any time during the interval and make a purchase decision based on the prevailing market prices, which ends the game. This substitutes for discounting.[17]

[15] The interpretation is that all products either remain homogeneous (e.g., a market for a certain book) or have known qualities q^i and production costs $c^i = q^i - 1$ (e.g., a market for branded ware).

[16] A reasonable assumption is that also a non-tracker, ultimately, observes the rival price and readjusts its own, e.g., if firm types reflect firms' unequal menu costs of tracking rivals and changing prices.

[17] Alternatively, we could assume that demand is uniformly distributed over the interval of interest and a measure of consumers purchases a product during each subinterval $dt = [t_1 - t_0] \subset [0, 1]$.

The following timeline describes the order of moves more specifically. (Non-)tracker i refers to firm i whose type turns to be a (non-)tracker. Rival firms are commonly indexed by j.

1. Time $t = 0$.
   (a) Firm and consumer types realize.
   (b) Non-trackers commit to prices p_0^i.
2. Times $t = \frac{1}{n}, \ldots, \frac{n-1}{n}$ (for n odd).
   (a) Prices become observable. Tracker 1 can reset its price in odd periods $t = \frac{k}{n}$, for $k = 1, 3, \ldots, n-2$. Tracker 2 can choose new price in even periods $t = \frac{k}{n}$, for $k = 2, 4, \ldots, n-1$.
   (b) A consumer enters the market with probability $\frac{1}{2}$. A shopper buys from the firm whose current price is the lowest in the market and a non-shopper chooses the firm at random.
   (c) If a consumer arrives, the game ends and payoffs realize. If a consumer does not arrive, the game proceeds to the next period.
3. Time $t = 1$. The game ends and zero payoffs realize.

Hence, while the *ex ante* expected market demand from $t = 0$ to $t = 1$ is one consumer, it is also possible that no consumer arrives at the market during $t \in [0, 1]$. This eventuality becomes increasingly likely as the game proceeds to late periods.

Trackers can change their prices alternately on a grid at $t \in \left\{\frac{1}{n}, ..., \frac{n-1}{n}\right\}$. We start from a discrete time pricing game, where $n = \infty$, and thereafter focus on the continuous limit, where $n \to \infty$.[18]

It should be noted that the timing reveals to a tracker whether its rival is a tracker or a non-tracker whereas a non-tracker chooses its price while it is uncertain about its rival. We characterize the equilibrium for a single pricing game over $[0, 1]$.

Extending the analysis is possible, but with certain notable caveats. The first issue is that if a firm plays against the same rival over multiple intervals, such as $[0, 1]$, $[1, 2]$, etc., then a non-tracker will obviously learn whether its rival is a tracker. This affects equilibrium pricing in the later price games.[19] In reality, uncertainty about rival type is likely to persist because of three reasons: (i) uncertainty about the timing of rivals' tracking technology adoption, (ii) possible new upgrades, and (iii) entry and exit of firms.

[18]Our discrete time analysis corresponds with the finite horizon and our continuous time analysis with the infinite horizon of related dynamic oligopoly price games where $t \in [0, \infty)$, such as Bergemann and Välimäki (2006).

[19]See the closing remarks for a discussion.

For instance, an earlier tracker who commits to prices for a duration of $\frac{1}{n}$ becomes a non-tracker in relative terms if its rival installs a tracking update that decreases its commitment time to $\frac{1}{nm}$. The second obvious problem is that non-trackers may collude in a repeated game, whereas our analysis only covers non-collusive price equilibria among non-trackers. This is interesting in its own right and provides a useful point of comparison to collusive behavior.

3 Duopoly equilibrium

To solve the model from the back to the beginning, we start by considering a game with two trackers, who choose prices for $t = \frac{1}{n}$ through $t = 1$ in Section 3.1. We proceed in Section 3.2 to a tracker's optimal pricing against a non-tracker, where the price remains fixed after $t = \frac{2}{n}$, and derive in Section 3.3 a non-tracker's optimal mixed price strategy, where the price commitment is made at $t = 0$. The unconditional distribution of tracker prices is derived in Section 3.4 and the division of equilibrium payoffs investigated further in Section 3.5.

3.1 Tracker's problem against tracker

Discrete time game. We focus on Markov-perfect equilibria where the state variable is the current rival price p^j, which remains fixed throughout $[t - \frac{1}{n}, t + \frac{1}{n})$.[20] In a discrete time price game, out-of-equilibrium-path behavior is well-defined, overruling incredible threats. We denote the continuation value of tracker i at time t by V_t^i.

The Bellman equation of a tracker can be defined as

$$V_t^i(p^j) = \max_{p_i} \begin{cases} \frac{1}{n}\frac{1-\mu}{2}p^i + \left(1 - \frac{1}{n}\right) V_{t+1}^1(p^i), & \text{for } p_i > p_j \\ \frac{1}{n}\frac{1}{2}p^i + \left(1 - \frac{1}{n}\right) V_{t+1}^1(p^i), & \text{for } p_i = p_j \\ \frac{1}{n}\frac{1+\mu}{2}p^i + \left(1 - \frac{1}{n}\right) V_{t+1}^1(p^i), & \text{for } p_i < p_j \end{cases}$$

with the terminal condition $V_t^i(p^j) = 0$ (i) for any $t > \bar{t}$ following the arrival of the consumer in the market at $t = \bar{t}$, and (ii) for any $t \geq 1$, possibly followed by new independent price commitments at $t = 1$.[21]

Note that, if a consumer arrives in the market during $[0, 1]$, a firm's demand varies between $\frac{1-\mu}{2}$ (only captives) and $\frac{1+\mu}{2}$ (also shoppers). Because the firm can obtain *at least* the profit of $\frac{1-\mu}{2}$ by holding its price constant at unity and *at most* $\frac{1+\mu}{2}p$ by setting a

[20]This is without loss because feasible prices remain unchanged throughout, and current payoff only depends on current prices and whether a consumer has arrived.

[21]The game can thus be thought of as repeated as long as interaction between subsequent games remains limited enough to prevent collusion among non-trackers.

lower price $p < 1$, we see that a firm has no incentive to undercut a rival price p^j that lies below the low price bound $\underline{p} = \frac{1-\mu}{1+\mu}$, as defined by

$$\frac{1+\mu}{2}\underline{p} = \frac{1-\mu}{2}.$$

This price cutoff becomes useful for characterizing optimal symmetric tracker behavior.

Lemma 1 *For sufficiently large n and small ϵ, the symmetric best response of a tracker is given by*

$$p^i(p^j) = \begin{cases} p^j - \epsilon, & \text{for } p^j > \underline{p} \\ 1, & \text{for } p^j \leq \underline{p} \end{cases}$$

in the final period $t = \frac{n-1}{n}$, and by

$$p^i(p^j) = \begin{cases} p^j - \epsilon, & \text{for } p^j \in (1-\mu, 1] \\ p^j - \epsilon, & \text{for } p^j \in (\underline{p}, 1-\mu] \\ 1, & \text{for } p^j \leq \underline{p} \end{cases}$$

in all other periods $\frac{1}{n} < t < \frac{n-1}{n}$.

We thus find that a tracker optimally undercuts the rival price (i) by the minimal recognized discount ϵ if the rival price exceeds $1-\mu$ and (ii) drops the price to the low price bound $\frac{1-\mu}{1+\mu}$ if the other price is at most $1-\mu$. The reason is that prices remain constant for two consecutive periods in our alternating offers game. Thereby, the profit from a marginal price discount, $\epsilon < p^j - \underline{p}$, is

$$\frac{1}{n}\frac{1+\mu}{2}(p^j - \epsilon) + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1-\mu}{2}(p^j - \epsilon)$$

because the rival offers a smaller price in the coming period. However, the profit from a large price discount, $p^j - \underline{p}$, is

$$\frac{1}{n}\frac{1+\mu}{2}\underline{p} + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1+\mu}{2}\underline{p}$$

since this price will not be undercut in the next two periods. A tracker therefore trades off the current profit loss $\frac{1+m}{2}(p^j - \epsilon - \underline{p})$ from captives and shoppers and the next period's profit loss from captives $\frac{1-m}{2}(p^j - \epsilon - \underline{p})$ to the next period's profit gain from shoppers $\mu(p^j - \epsilon - \underline{p})$. As a result, a firm becomes more willing to drop its price to $\underline{p}$ as the rival price p^j decreases over time. Shoppers hasten the timing. Figure 2 describes this price cycle as an automation, thus demonstrating the potential for automated implementation, as in Brown and MacKay (2023).

Because firms have captive demand, our derived cycle differs from the standard Edge-

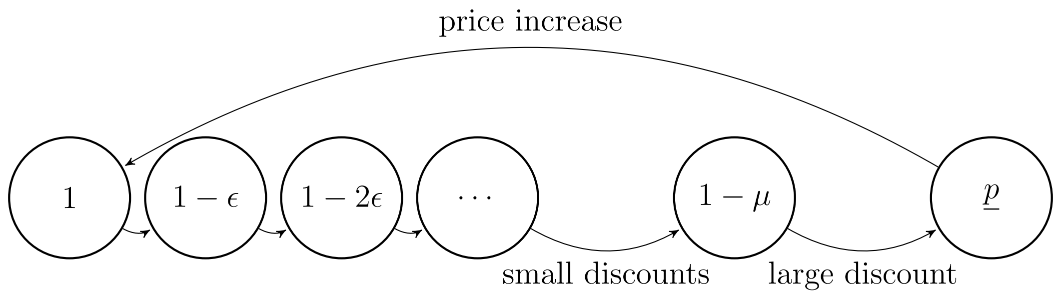


Figure 2: Price cycle automation in the subgame between two trackers.

worth cycle of Maskin and Tirole (1988) in that prices are initially cut by the minimum amount but drop directly to the bottom from a mid-level, thereafter, followed by the usual lenience phase in which prices bounce to the monopoly level and the cycle restarts.

As profits increase in current prices, which imply higher prices in the future – Lemma 2 below – we find that a tracker optimally undercuts its rival by as small a discount as possible if the rival price exceeds $1-\mu$. Moreover, to commit both firms to as high future prices as possible, tracker 1 chooses the monopoly price in the starting period $t = 1$.

Lemma 2 $V_t^i(p^j)$ *is increasing in* p^j *for any* $\frac{2}{n} < t < \frac{n-1}{n}$ *and any* $1-\mu < p^j < 1$.

Consistent with existing literature, we take ϵ as zero and suppose as a tie-break rule that shoppers buy from the most recent firm to alter its price.[22] A unique price equilibrium hence arises, prices being close to the monopoly level.

Proposition 1 *In a discrete time price game, the unique equilibrium price sequence equals* $(1, 1-\epsilon, 1-2\epsilon, ..., 1-n\epsilon)$. *As* $n\epsilon \to 0$, *the expected tracker profit is* $\frac{1}{2}$ *and the expected consumer surplus* 0.

Because the consumer is equally likely to arrive in odd and even time periods, both firms have the same probability of winning any shoppers in this case. Each tracker thus obtains the profit of $\frac{1}{2}$, leaving no surplus to consumers.

Continuous time game. As the time between price adjustments approaches zero, new collusive price patterns emerge with appropriate penalties. The hardest penalty would be reverting to $\underline{p}$ for the rest of the game if the rival deviates from the collusive price. The rival would then respond by setting the monopoly price, thus limiting both firms to the standard duopoly profits, i.e., $\frac{1-\mu}{2}1 = \frac{1+\mu}{2}\underline{p}$.[23] However, here this punishment lacks credibility because the penalizing firm would have an incentive to undercut the rival. Instead, the equilibrium price cycle in Figure 2 turns to be an applicable punishment, started from a lowish price.

Lemma 3 (Folk theorem) *In a continuous time price game, any price sequence where*

[22]We thank Andrew Rhodes for this insight. See the Appendix for limit order analysis of ϵ and n.
[23]This idea appears in (Myatt and Ronayne, 2019) where firms commit not to undercut list prices.

firms' continuation payoffs exceed $(1-t)\frac{1-\mu}{2}$ *at* t *can be implemented in equilibrium for* $n\epsilon \to 0$.

In particular, any collusive price $p \in (1-\mu, 1)$ is sustained under the threat of reverting to a punishment cycle started at a lower price $p \in (1-\mu, 1)$. The start price of the punishment cycle matters. By starting the cycle at price $p > 1-\mu$ at time t, firms acquire the payoffs $(1-t)\frac{p}{2}$. Firms keep undercutting each other over the cycle but, as $n\epsilon \to 0$, the price level p barely budges over $[t, 1]$.

Proposition 2 *In a continuous time price game, maximum tracker payoffs* $\frac{1}{2}$ *and minimal consumer surplus* 0 *obtain in a collusive equilibrium where firms set the monopoly price.*

Proof. A sketch of proof is above. □

It is noteworthy that trackers' high profits cannot be wholly attributed to collusive price behaviour. Comparing Propositions 1 and 2, we can see that equilibrium profits remain high both in the discrete time game, where the price pattern is cyclic, and in the continuous time game, where firms set collusive prices. This observation is reminiscent of Brown and MacKay (2023) who find that automated price setting leads to supra-competitive prices without collusion. Here this happens through a sequence of negligible price concessions and, in their model, because prices reside between the Bertrand and Stackelberg levels.

3.2 Tracker's problem against non-tracker

A tracker's problem against a non-tracker turns out to be much simpler. Because a non-tracker cannot retaliate, a tracker undercuts a non-tracker by the minimum discount ϵ if the price of the non-tracker exceeds the low price bound $\underline{p}$.

Clearly, by setting the monopoly price unity, a tracker receives the expected profit

$$\pi^i = \frac{1-\mu}{2}$$

from non-shoppers who observe its price but fail to observe the price of its rival. Instead, by undercutting the rival price p^j by the minimum recognized discount ϵ, a tracker obtains

$$\pi^i = \left(\frac{1-\mu}{2} + \mu\right)(p^j - \epsilon)$$

by selling to its captives and to shoppers who compare its price $p^j - \epsilon$ to the rival p^j. Thus, a tracker strictly benefits from undercutting the price of a non-tracker iff $p^j - \epsilon > \underline{p}$.

Note that, because a non-tracker commits to its price for the rest of the game, a rival tracker has no reason to change its price after setting it. In a market with a tracker i and a non-tracker j, optimal prices thus remain constant over $[\frac{i}{n}, 1]$.

3.3 Non-tracker's problem

A non-tracker chooses its price without knowing whether it faces a tracker or another non-tracker. Because a non-tracker does not follow the market but sporadically, it is important for it to account for uncertainty about rival types. The prospect of facing another non-tracker makes it optimal for a non-tracker to price in mixed strategies. In general, the payoff to a non-tracker for setting a price above $\underline{p}$ can be expressed as

$$\pi^i = \begin{cases} \left(\frac{1-\mu}{2} + \mu(1-\tau)(1-F(p^i)) + \frac{\mu(1-\tau)}{2}\mathbf{1}\left\{p = p^j\right\}(p^i)\right) p^i, & \text{if } p^i > \underline{p}, \\ \left(\frac{1-\mu}{2} + \mu\right) p^i, & \text{if } p^i = \underline{p}, \end{cases} \tag{1}$$

presuming that the price of another non-tracker follows the distribution F (F denotes the cdf and f the related pdf; $\mathbf{1}\left\{p = p^j\right\}(p)$ indicates a potential tie).

The main difference from the usual Varian (1980) model is that here a tracker is presumed to undercut any price $p^i > \underline{p}$. As a result, a non-tracker can only sell to a shopper (i) by setting a (mixed) price $p^i > \underline{p}$ that lies below the price of a rival non-tracker, or (ii) by setting a (limit) price $p^i = \underline{p}$ that a rival tracker is not willing to undercut. It follows that the lowest price $p' > \underline{p}$ that a non-tracker applies in competition with another non-tracker is given by

$$\pi^i = \left(\frac{1-\mu}{2} + \mu(1-\tau)\right) p' \Longrightarrow p' = \frac{1-\mu}{2(1-\tau)\mu + 1 - \mu} \in \left(\frac{1-\mu}{1+\mu}, 1-\mu\right). \tag{2}$$

The support of the equilibrium price distribution thus contracts from the standard $[\underline{p}, 1]$ to $[p', 1]$ due to the presence of trackers. The intuition is that the lowest prices are no longer optimal for a non-tracker because uncertainty about the rival type renders its demand less elastic. In the intermediate price range $(\underline{p}, 1)$, a non-tracker can only win against a non-tracker. Without knowing its rival's type, a non-tracker thus becomes unwilling to offer the lowest prices $(\underline{p}, p')$.

Following the standard logic as in Varian (1980) with the updated demand function, we therefore find that a non-tracker's pricing in a randomized equilibrium is given by Eq. (3)

$$F(p^i) = 1 + \frac{1}{2\mu}\frac{1-\mu}{1-\tau}\left(1 - \frac{1}{p^i}\right), \tag{3}$$

for $p^i \in [p', 1]$, where $p' = \frac{1-\mu}{1+\mu-2\mu\tau} > \underline{p} = \frac{1-\mu}{1+\mu}$. The expected price is

$$E(p^i) = \frac{1-\mu}{2(1-\tau)\mu} \ln\left(1 + \frac{2(1-\tau)\mu}{1-\mu}\right), \tag{4}$$

and the expected profit $\pi^i = \frac{1-\mu}{2}$.[24]

Note that, although a non-tracker's expected price increases with more trackers, the expected profit remains unchanged as the number of trackers increases. Interestingly, we thus observe that wider presence of trackers in a market redistributes payoffs from consumers to trackers but leaves the payoff division between consumers and non-trackers unaltered. Figures 3.4a and 3.4b depict the changes in F with respect to μ and τ. As can be seen in Eq. (3), a higher number of trackers (a lower number of shoppers) increases the prices of non-trackers F in the sense of first-order stochastic dominance.

Notably, the mixed equilibrium described above is the unique firm-type-symmetric equilibrium in this case given that the limit-price equilibrium which arises later with three firms fails to exist in a duopoly.[25] In the discussed limit-price equilibrium, a non-tracker would commit to $\underline{p}$ and a tracker would respond by monopoly pricing. A profitable deviation arises because another non-tracker could also offer the same low price. A deviation to 1 would thus yield a higher tracker profit than the commitment to $\underline{p}$, which gives $\left(\frac{1-\mu}{2} + \mu\right) \underline{p} = \frac{1-\mu}{2}$ against a tracker but $\left(\frac{1-\mu}{2} + \frac{\mu}{2}\right) \underline{p} < \frac{1-\mu}{2}$ against a non-tracker.[26]

It is also noteworthy that a tracker can observe the type of the rival before price setting. This simplifying assumption allows us to abstract from learning stage that would otherwise ensue. Our game allows a tracker to learn the rival type from either the timing of price changes or its response to other prices.[27] This implies that a non-tracker cannot benefit from committing to the collusive price that trackers apply in the hope that a tracker would mistake it for another tracker. We can hence proceed under the assumption that a tracker undercuts a non-tracker and sets the monopoly price with another tracker.

3.4 Tracker's pricing

Here, tracker prices inherit the appearance of randomness from the non-tracker prices that they follow, although only non-trackers price in mixed strategies while trackers apply pure price strategies. If the rival is a non-tracker, tracker prices follow F and, if the rival is a tracker, the tracker price equals unity. Unlike the standard case, the equilibrium distribution of tracker prices G now becomes a mixture $G(p) = (1-\tau)F(p) + \tau \mathbf{1}_{\{p\geq 1\}}(p)$ of a continuous discount distribution F and an indicator (step) function $\mathbf{1}_{\{p\geq 1\}}$, representing

[24] Note for later use that the cdf of the minimum of two non-tracker prices is $F_{(1)}(p) = 1-(1-F(p))^2 = 1 - \frac{1}{4\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(1-\frac{1}{p}\right)^2$ and the pdf $f_{(1)}(p) = 2(1-F(p))f(p) = -\frac{1}{2\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(1-\frac{1}{p}\right)\frac{1}{p^2}$, which gives the average minimum price $E(p_{(1)}) = \frac{1}{2\mu^2}\frac{(1-\mu)^2}{(1-\tau)^2}\left(\ln p' + \frac{1}{p'} - 1\right)$.

[25] Following up the case of three firms is thus necessary to uncover the most interesting price equilibria.

[26] Baye et al. (1992) show for the Varian (1980) *model of sales* that, while a unique symmetric equilibrium arises in a duopoly, for more than two firms, there also exist a continuum of asymmetric equilibria. We find multiple type-symmetric equilibria for triopolies.

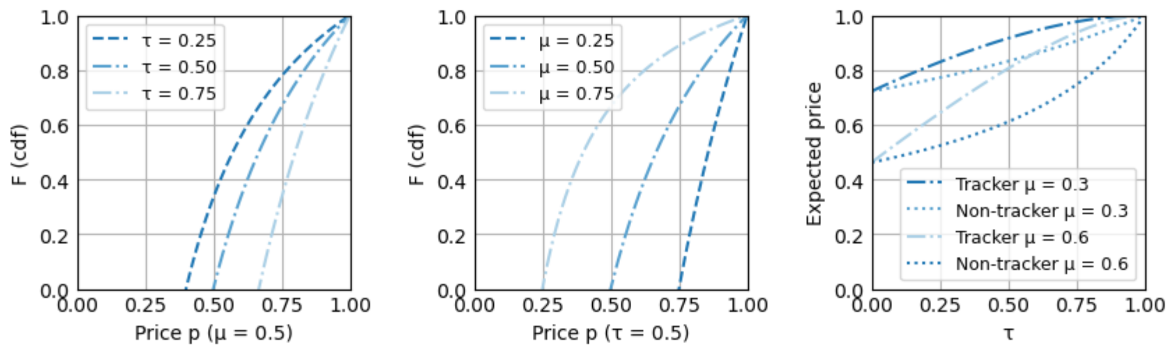
[27] Incorporating a learning phase to the current model is easy but complicates the analysis without offering much in return. In the interest of brevity, we thus concentrate here on post-learning behavior, adding a couple words about learning in the conclusion.

a mass point at price one. The strategy yields a tracker the expected profit of

$$\tau\frac{1}{2} + (1-\tau)\frac{1+\mu}{2}E(p) = \tau\frac{1}{2} + \frac{1-\mu^2}{4\mu}\ln\left(\frac{2(1-\tau)\mu + 1 - \mu}{1-\mu}\right).$$

This profit exceeds the typical payoff $\frac{1-\mu}{2}$ of a non-tracker for two reasons. First, when a tracker competes against another tracker, both earn collusive profits in continuous pricing. Second, when a tracker faces a non-tracker, a tracker wins shoppers with probability one by undercutting its rival by the minimal discount ϵ although both a tracker's and a non-tracker's prices outwardly follow the same distribution function F. A non-tracker only wins against a non-tracker and, even then, has a smaller winning probability, $1-F(p^i) < 1$ for $p^i > p'$. Figure 3c compares the expected prices of trackers and non-trackers.

It is thus worth emphasizing that price dispersion in our model does not fully rely on randomizes price strategies, unlike in models in the spirit of Varian (1980); Stahl (1989); Burdett and Judd (1983). Here only non-trackers price in mixed strategies whereas trackers simply follow them. To an outside observer or economist, both prices nevertheless look random, i.e., the unconditional price distribution is non-degenerate. This subtle point might be crucial for developing an good understanding of the morphology of equilibrium price dispersion, studied in, for example, Kaplan and Menzio (2015).



(a) $F(p)$ (b) $F(p)$ (c) $E(p)$

Figure 3: Prices

3.5 Division of payoffs

The consumer surplus of a non-shopper is obtained as follows

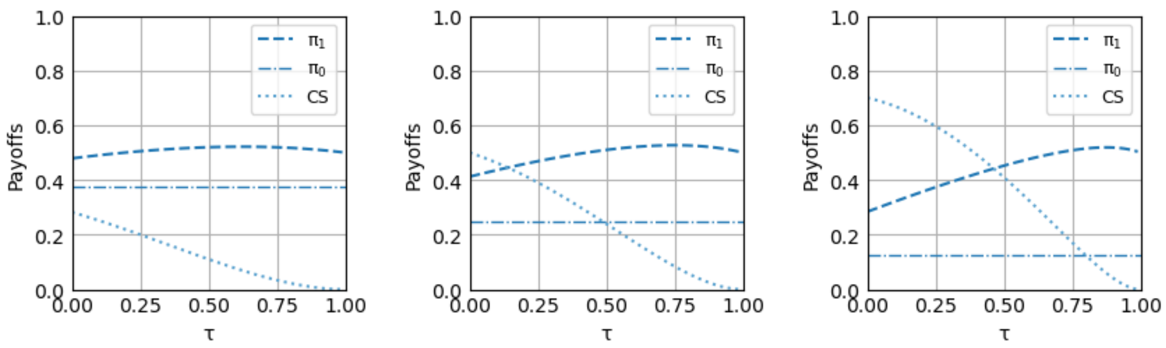
$$CS_0 = 2\tau(1-\tau)\left(\frac{1-E(p_0)}{2} + \frac{1-E(p_0)+\epsilon}{2}\right) + (1-\tau)^2\left(1-E(p_0)\right),$$

whereas the consumer surplus to a shopper can be expressed as

$$CS_1 = 2\tau(1-\tau)\left(\frac{1-E(p_0)}{2} + \frac{1-E(p_0)+\epsilon}{2}\right) + (1-\tau)^2(1-E(p_{(1)}),$$

where $E(p_0)$ denotes the expected non-tracker price and $E(p_{(1)})$ the expected minimum of two non-tracker prices.

Non-shoppers and shoppers alike thus face higher prices in markets where trackers collude. Furthermore, in markets where both a tracker and a non-tracker are present at the same time, shoppers are burdened with higher prices as the minimum price becomes $E(p_0) - \epsilon$, whereas in markets without trackers, the minimum price is lower at $E(p_{(1)})$. Consequently, the expected consumer surplus, which is calculated as $\mu CS_0 + (1-\mu)CS_1$, falls below the usual μ.



(a) $\mu = 0.25$ (b) $\mu = 0.50$ (c) $\mu = 0.75$

Figure 4: Payoff effects of τ

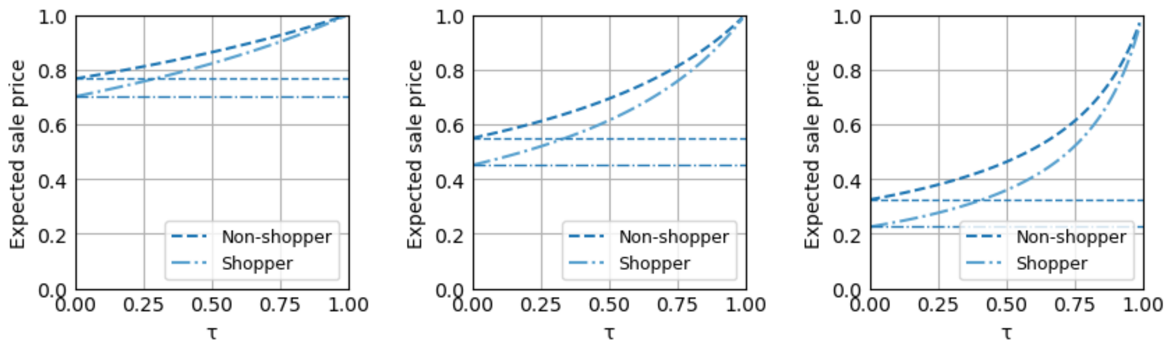
To summarize, the presence of trackers in the current duopoly setup always reduces consumer surplus since trackers receive higher profits while the profit of a non-tracker remains unchanged.[28] Figure 4 illustrates the impact of trackers on payoffs. It should be noted that the payoffs do not add up to unity for $\tau \neq \frac{1}{2}$ as the number of firms receiving tracker payoffs π_1 increases and the number of firms receiving non-tracker payoffs π_0 decreases as τ increases.

Interestingly, trackers only benefit from the presence of other trackers up to a certain point, after which their payoffs start to decline with the increasing number of competing trackers. This is because collusion becomes less profitable if the competitor is more likely to be a tracker. The mechanism is novel to the literature. When a tracker competes with another tracker, it obtains a higher collusive price of 1, but captures only half the shoppers. However, if a tracker is competing with a non-tracker, it receives a lower expected price of $E(p_0)$ but attracts all the shoppers. Since a non-tracker offers higher prices if the competitor is more likely a tracker, the difference $1 - E(p_0)$ between the collusive price and non-tracker price is decreasing in τ. As $1 - E(p_0) \to 0$ as $\tau \to 1$, the negative payoff effect of sharing shoppers with another tracker dominates the positive payoff effect of collusion for high enough values of τ.

Likewise, the wider presence of trackers also harmonizes prices paid by shoppers and non-shoppers for two reasons. Firstly, there is the previously explained convergence be-

[28]We show in the next section that all results need not hold in a larger market.

tween trackers' and non-trackers' expected prices. Secondly, if a tracker and a non-tracker are both present, the non-tracker prices at $E(p_0)$ and the tracker at $E(p_0) - \epsilon$, thus offering the consumer only a slightly lower price than its rival. This price-unifying effect of tracker presence eliminates opportunities for price arbitrage benefiting shoppers when trackers are abundant. These results suggest that a more extensive tracker presence may reduce the number of shoppers in the long run, but not necessarily harm consumer welfare if market prices are unified. Figure 5 shows the effect of trackers on the expected gap between shopper and non-shopper prices.



(a) $\mu = 0.25$ (b) $\mu = 0.5$ (c) $\mu = 0.75$

Figure 5: Sale price effects of τ

4 Triopoly equilibrium

More elaborate price patterns arise in a larger market where firms differ in commitment power and consumers in price information. To analyse such markets with $m > 2$ firms, we assume, as before, that all non-trackers commit to prices at time $t = 0$ while trackers take turns in setting new prices, i.e., tracker i can reset its price at times $t = \frac{i+m}{n}$ for $m = 0, 1, ...$ The number of trackers is denoted by r. We concentrate here on triopoly markets, which suffice to show the main insights. The analysis extends immediately to larger market environments.[29]

4.1 Tracker and non-tracker cutoffs

We proceed to characterize an equilibrium by starting from the assumption that trackers undercut the lowest price chosen by a non-tracker $\tilde{p} := \min_i p_0^i$ if the price lies above a cutoff $\underline{p} := \underline{p}_r$, which can depend on the number of trackers $r \leq 3$ that the tracker can observe. Extending our earlier analysis, we can see that trackers can extract from

[29]We have studied also markets with more firms, which allow for more complex price patters, e.g., a number of gaps in the equilibrium price distribution and several profit-ranked limit-price equilibria.

shoppers at most the joint residual profit $\mu\tilde{p}$ by either setting a collusive price $\tilde{p} - \epsilon$ or implementing a price cycle that starts from $\tilde{p} - \epsilon$.[30] To maximize the payoffs of trackers, we concentrate on the former hereone.

In a market with three firms, a tracker obtains the profit of $\frac{1-\mu}{3}$ by selling to captives at price $p = 1$. Collusion among trackers at price $\tilde{p} - \epsilon$ provides thereby a higher tracker payoff if

$$\left(\frac{\mu}{r} + \frac{1-\mu}{3}\right)\tilde{p} \geq \frac{1-\mu}{3} \Longrightarrow \tilde{p} \geq \frac{1-\mu}{\frac{3}{r}\mu + 1 - \mu} =: \underline{p}_r,$$

as $\epsilon \to 0$. Because firms have fewer captives now, the price bound for a single tracker, $\underline{p}_1 = \frac{1-\mu}{1+2\mu}$, is smaller in a triopoly than in the earlier studied duopoly, where the price cutoff was $\underline{p} = \frac{1-\mu}{1+\mu}$. Because shoppers are shared among a larger number of trackers under collusion, the price bound is increasing in r. For multiple trackers $r > 1$, the price bound $\underline{p}_r$ clearly exceeds $\underline{p}$. showing as standard that collusion becomes more difficult to sustain among more firms.[31]

By contrast, the optimal pricing behavior of a non-tracker depends on (i) how many trackers are present in the market and (ii) the smallest prices that the trackers will undercut. A non-tracker only knows that the total number of rivals it faces is two, but does not know how many are trackers. By and large, a single tracker tends to be more willing to undercut a price than two trackers who share the profit from shoppers. As the *expected* behavior of trackers determines the optimal strategy of non-trackers, various pricing patterns might arise.

As a first guess, we presume that trackers always undercut the lowest price among non-trackers, denoted hereon by $\tilde{p}$. Because a non-tracker only wins shoppers if both the other firms are non-trackers, the lower bound on the prices that a non-tracker is willing to offer is given by

$$\left(\frac{1-\mu}{3} + (1-\tau)^2\mu\right)p_2'' = \frac{1-\mu}{3} \Longrightarrow p_2'' = \frac{1-\mu}{3(1-\tau)^2\mu + 1 - \mu}.$$

However, as the number of trackers in the market increases, a tracker may not be willing to undercut $\tilde{p}$ because the profits from collusion at price $\tilde{p} - \epsilon$ are shared among a larger group of trackers. If only a lone tracker is expected to undercut $\tilde{p}$, the cutoff for prices that a non-tracker offers reduces to

$$\left(\frac{1-\mu}{3} + \tau^2\mu + (1-\tau)^2\mu\right)p_1'' = \frac{1-\mu}{3} \Longrightarrow p_1'' = \frac{1-\mu}{3(1-\tau)^2\mu + 3\tau^2\mu + 1 - \mu}.$$

Similarly, if trackers never undercut p'', trackers only offer prices above

[30] The price cycle in Figure 2 serves as a punishment to sustain collusion in both triopolies and duopolies; see the Appendix.

[31] Our investigations into a four-firm market show that sustaining collusion becomes increasingly difficult in a larger market.

$$p_0'' = \frac{1-\mu}{3\mu + 1 - \mu}.$$

The above cutoffs cover all the possibilities in a market with three firms. Because $p_0'' < p_1'' < p_2''$, we can see that non-trackers are willing to offer lower prices if they expect that trackers are less likely to undercut them. In equilibrium, non-tracker beliefs must coincide with tracker behavior – Lemma 4.

Lemma 4 *If the sup of a non-tracker's prices equals one, an equilibrium p'' satisfies either of the following options*

1. $\underline{p}_1 < \underline{p}_2 < p'' = p_2'' \Longleftrightarrow \tau > 1 - \frac{1}{\sqrt{2}}$
2. $\underline{p}_1 < p'' = p_1'' \leq \underline{p}_2 \Longleftrightarrow \tau > 0.$

Lemma 4 shows, on the one hand, that if non-trackers *believe* that colluding trackers will undercut any price offer they make, they will do so for $\tau > 1 - \frac{1}{\sqrt{2}}$. A higher expected number of trackers discourages a non-tracker from making large price concessions because it only wins against another non-tracker. This decreases the non-tracker cutoff p_2'', allowing it to surpass the tracker cutoff $\underline{p}_2$. On the other hand, Lemma 4 proves that, if non-trackers *expect* that only a single tracker is willing to undercut their lowest price, this will similarly hold true for any $\tau > 0$. A non-tracker is willing to reduce its lowest prices p_1'' if the price is never undercut by colluding trackers. As p_1'' hence decreases below $\underline{p}_2$, trackers indeed abandon collusion for monopoly pricing.

4.2 Tracker and non-tracker prices

Thanks to Lemma 4, the existence of an equilibrium is not a problem in the markets we analyze. On the contrary, we observe that multiple equilibria with distinctive price dynamics exist. The primary reasons for the multiplicity can be pinpointed in the fact that the prices of trackers are (i) *strategic complements* for high non-tracker prices, which the trackers are willing to undercut, but (ii) *strategic substitutes* for lower non-tracker prices; the demarcating cutoff price depends on the competition between trackers. Instead, the prices of non-trackers are invariably (iii) *strategic substitutes* to tracker prices, in the sense that non-trackers set lower prices if these prices are not undercut by two trackers. We next discuss each tentative equilibrium in turn.

Alternative I: Mixing below the monopoly price

The simplest equilibrium pattern arises when trackers are abundant.

Proposition 3 (Alternative I: random high prices) *If* $\tau > 1 - \frac{1}{\sqrt{2}}(\approx 0.2929)$, *there exists a market equlibrium where non-trackers mix across* $[p_2'', 1]$ *while trackers offer* $\tilde{p} - \epsilon$ *for* $\tilde{p} \geq p_2''$.

This benchmark perhaps most resembles our previous duopoly equilibrium. Non-trackers expect trackers to undercut them regardless of how many of them there are. This reduces a non-tracker's incentive to offer lower prices, reinforcing a tracker's incentive to undercut non-trackers. More precisely speaking, existence requires that $\frac{\mu}{2} \geq (1-\tau)^2\mu$, i.e., that a tracker can expect more demand from shoppers in collusion with a single other tracker (lhs) than the demand from shoppers that a non-tracker expects from offering the lowest price among non-trackers (rhs).[32]

Alternative II: Mixing below the monopoly price and a limit price

In an alternative equilibrium, non-trackers use either lower or higher prices but not intermediate ones. Trackers undercut the higher prices but not the lower, to which they respond by setting the monopoly price. Thus, the equilibrium price distribution has a gap and an atom.[33]

Proposition 4 (Alternative II: Random high and low prices) *For any* $\tau < \frac{1}{\sqrt{2}}$, *there exists an alternative equilibrium where non-trackers mix across* $[p_1'', \underline{p}_2]$ *and* $[\underline{p}_2 + a_2, 1]$. *A single tracker offers* $\tilde{p} - \epsilon$ *and a duo of two trackers* $\tilde{p} - \epsilon$ *for* $\tilde{p} > \underline{p}_2$ *and* 1 *for* $\tilde{p} \leq \underline{p}_2$.

The presence of a gap is quite an uncommon model property, valuable for improving the fit between the model and reality.[34] Here the gap in the equilibrium price distribution arises because the prices directly above the cutoff $\underline{p}_2$ provide strictly less profit to non-trackers than the prices directly below.

In particular, by pricing at $\underline{p}_2 + \epsilon$, a non-tracker obtains

$$\left(\frac{1-\mu}{3} + \mu(1-\tau)^2(1-F(\underline{p}_2))^2\right)(\underline{p}_2 + \epsilon)$$

because it competes with non-trackers and is always undercut by trackers.

By comparison, by pricing at $\underline{p}_2 - \epsilon$, a non-tracker extracts

$$\left(\frac{1-\mu}{3} + \mu\left(\tau^2 + (1-\tau)^2(1-F(\underline{p}_2))^2\right)\right)(\underline{p}_2 - \epsilon)$$

because it competes with non-trackers but is not undercut by two trackers.

[32]Equilibrium price distributions are derived in the Appendix.

[33]A more complex variant of this has more gaps: Non-trackers use either lower, intermediate, or higher prices but not certain prices in between. All trackers undercut the highest prices. The prices in the middle are, however, at the same time, (i) so low that a larger group of trackers is not willing to undercut them, and (ii) so high that a smaller tracker group finds it worthwhile to underbid. The lowest prices are only undercut by a single tracker. This equilibrium needs at least four firms.

[34]See Ellison and Wolitzky (2012) for another example of a gap.

To compensate for the difference in profits, only prices lying above $\underline{p}_2+a_2$ are employed by non-trackers in equilibrium. The gap size $a_2 > 0$ is defined by

$$\left(\frac{1-\mu}{3}+\mu(1-\tau)^2(1-F(\underline{p}_2))^2\right)(\underline{p}_2+a)=\left(\frac{1-\mu}{3}+\mu\left(\tau^2+(1-\tau)^2(1-F(\underline{p}_2))^2\right)\right)\underline{p}_2$$

We find that $\underline{p}_2 + a_2 < 1$ iff $\tau < \frac{1}{\sqrt{2}}$.

Alternative III: Mixing below a limit price

The ability to sustain a gap above the tracker price cutoff, $[\underline{p}_2, \underline{p}_2 + a_2]$, brings up another interesting possibility. If non-trackers can commit to pricing below the tracker price cutoff, $\underline{p}_2$ or $\underline{p}_1$, they can significantly reduce their exposure to tracker competition. The idea is akin to limit pricing, where incumbents employ low prices to deter entry. To investigate this possibility, we next suppose that the highest price employed by a non-tracker in equilibrium is $\underline{p}_2$ or $\underline{p}_1$in stead of unity.

As can be seen, pricing at $\underline{p}_2 - \epsilon$, for $\epsilon \to 0$, yields more profit to a non-tracker per period than a deviation to unity if

$$\left(\frac{1-\mu}{3}+\mu\tau^2\right)\underline{p}_2 \geq \frac{1-\mu}{3} \iff \underline{p}_2 \geq \frac{1-\mu}{3\mu\tau^2+1-\mu}.$$

This condition is satisfied iff $\tau \geq \frac{1}{\sqrt{2}}$. In the same vein, pricing at $\underline{p}_1 - \epsilon$, for $\epsilon \to 0$, yields more profit to a tracker per period than a deviation to unity if

$$\left(\frac{1-\mu}{3}+\mu\tau^2+2\mu\tau(1-\tau)\right)\underline{p}_1 \geq \frac{1-\mu}{3} \iff \underline{p}_1 \geq \frac{1-\mu}{3\mu\tau^2+6\mu\tau(1-\tau)+1-\mu},$$

which is only satisfied under condition $\tau = 1$, i.e., in a market where no non-trackers exist. We can thus focus on cases with the limit price $\underline{p}_2$.[35]

Lemma 5 *If the sup of non-trackers' prices equals $\underline{p}_2$, an equilibrium p'' satisfies $\underline{p}_1 < p'' = p_1''(\underline{p}_2) < \underline{p}_2$, which holds for all $\tau > \frac{1}{\sqrt{2}}$.*

We thus find that $p'' = p_0'' \leq \underline{p}_1$ is not feasible. This is natural because a single tracker, whose profit from captives remains $\frac{1-\mu}{3}$, is willing to offer lower prices to attract shoppers than a non-tracker, whose expected profit surpasses $\frac{1-\mu}{3}$ in this case.

$$\left(\frac{1-\mu}{3}+\mu\tau^2\right)\frac{1-\mu}{\frac{3}{2}\mu+1-\mu}$$

Proposition 5 (Alternative III: Random low prices) *For $\tau \geq \frac{1}{\sqrt{2}}(\approx 0.7071)$, there exists an alternative equilibrium where non-trackers mix across $[p_1''(\underline{p}_2), \underline{p}_2]$. A single tracker*

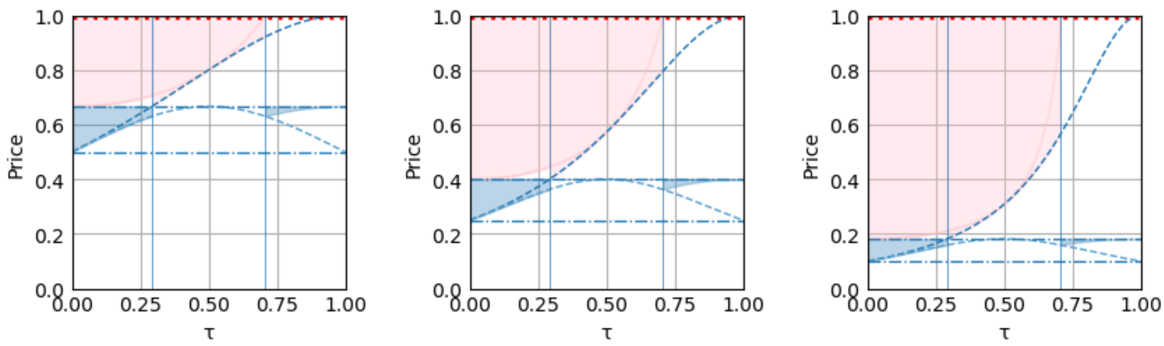
[35] In a larger market, the limit-price may not be unique. Non-trackers can price either below a higher price, e.g., $\underline{p}_4$ or $\underline{p}_5$, that a larger group of trackers will not undercut, or below a lower price, e.g., $\underline{p}_2$ or $\underline{p}_3$ which deters collusion by a smaller tracker group.

offers $\tilde{p} - \epsilon$ and a duo of two trackers 1.

Our described limit-price mechanism to restrict collusion is different from earlier ones where collusion is made unstable by firm-side heterogeneity, such as network variation in Clark and Houde (2013) or asymmetric capacities in Fonseca and Normann (2008).[36] Here colluding firms are homogenous while the non-colluding rivals intentionally disturb collusion that hurts them. The presence of non-trackers also limits collusive profits in the alternative equilibrium patterns.

Because non-tracker profit is maximized under limit pricing, we would like to argue that the equilibrium is focal in the sense that the firms who start the game would like to coordinate to this equilibrium (Mailath et al., 1993). Thus, while multiple equilibria exist for $\tau > 1 - \frac{1}{\sqrt{2}}$, payoff-dominance selects the limit-price equilibrium for $\tau \geq \frac{1}{\sqrt{2}}$ and simplicity favors the equilibrium without a gap for $1 - \frac{1}{\sqrt{2}} < \tau < \frac{1}{\sqrt{2}}$.[37] For $\tau \leq 1 - \frac{1}{\sqrt{2}}$, a unique equilibrium exists.

Figure 6 depicts the supports of equilibrium price distributions (filled) presuming this selection for varying numbers of shoppers and trackers. Two trackers are willing to collude below non-tracker prices in the pink regions but only a single tracker undercuts non-tracker prices in the blue regions. Dashdot lines denote $\underline{p}_1$ and $\underline{p}_2$ and dashed lines p_1'' and p_2''. Red dots represent atoms at the monopoly price. Atoms arise for all τ because three trackers have nothing to undercut.



(a) $\mu = 0.25$ (b) $\mu = 0.50$ (c) $\mu = 0.75$

Figure 6: The supports of equilibrium price distributions.

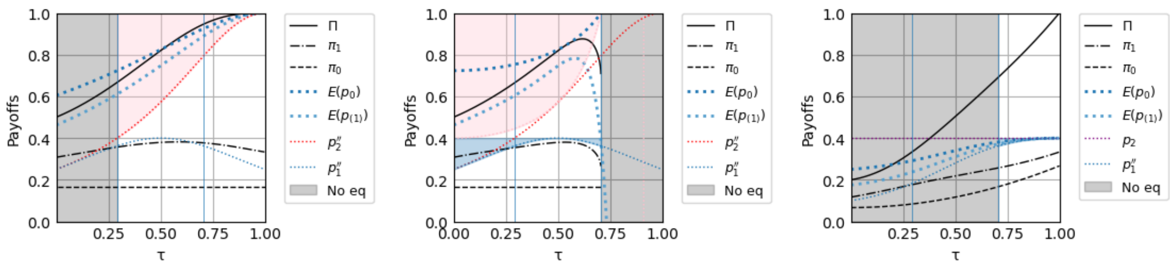
Looking at Figure 6, the gap $[\underline{p}_2, \underline{p}_2 + a_2]$ (the tiny unfilled region above a line) appears quite small in Alternative II ($\tau < 1 - \frac{1}{\sqrt{2}}$) and so too does the support $[p_1''(\underline{p}_2), \underline{p}_2]$ (the tiny filled region below a line) in Alternative III ($\tau > \frac{1}{\sqrt{2}}$). The empirical implication is that we should expect to see a lot of variation in market prices if trackers are few. However, as trackers become increasingly common in the market, the probability mass will concentrate

[36] Introducing firm heterogeneity as in Narasimhan (1988) to our model would be straightforward. The firms with the smallest consumer base would then be the most tempted to undercut rivals or collude.

[37] The length of the second interval $[p_1'', \underline{p}_2]$ approaches zero as τ approaches $\frac{1}{2}$, which implies that the prices lying below the gap are rarely applied for intermediate values of τ in the "gap" equilibrium.

around $\left\{\underline{p}_2, 1\right\}$. The empirical price distribution should thus look almost bimodal, with a significant gap between modal prices, in a market where trackers are prevalent.

4.3 Tracker and non-tracker profit



(a) Alt I (b) Alt II (c) Alt III

Figure 7: Industry profits Π and expected profits π_1 and π_0 ($\mu = 0.5$).

Figure 7 shows industry profits and the expected profits of trackers and non-trackers under the three alternative equilibria.[38] A general feature is that profits increase and consumer surplus thereby decreases with more trackers if the number of trackers is either small $\tau < 1 - \frac{1}{\sqrt{2}}$ or already large $\tau > \frac{1}{\sqrt{2}}$. Because the presence of trackers alleviates competition among non-trackers, non-trackers offer higher prices. The trackers who undercut these prices will therefore also charge higher prices, making high prices more salient. The expected profit of colluding at monopoly prices in an all-tracker market is obviously also increasing in τ.

Nevertheless, the difference between tracker and non-tracker profit is generally non-monotone in the number of trackers. A higher τ implies that a firm is more likely to face another tracker than two non-trackers. The profit of a single tracker against two non-trackers is $\frac{1+2\mu}{3}E(p_{(1)})$ but the profit of a tracker against a tracker and a non-tracker is either (i) $\frac{2+\mu}{3}E(p_0)/2$, if the trackers collude and divide the shoppers, or (ii) $\frac{1-\mu}{3}$, if the trackers sell to their captives only. The non-tracker profit is either constant or increases with more trackers. The benefit of being a tracker is therefore first increasing in τ but thereafter decreasing.

In the gap-equilibrium the non-monotone pattern is particularly pronounced below the upper end of the support $\tau = \frac{1}{\sqrt{2}}$. This is because the gap increases relatively quickly and opportunities for profitable collusion therefore shrink rapidly. In the neighborhood of $\tau = \frac{1}{\sqrt{2}}$, non-trackers mostly offer prices below $\underline{p}_2$, which prevents two trackers from undercutting them.

[38]See the Appendix for the details.

5 Conclusion

We demonstrate how heterogeneous price technologies on both market sides affect equilibrium price distribution. This involves marrying the static Varian (1980) model of sales with Maskin and Tirole (1988) for a dynamic extension.[39] We observe that consumers are generally hurt by the presence of firms whose price commitments have a shorter duration. The negative price effects of tracking are shown to be broad and not solely collusion-related. The impact on consumer search incentives merits further elaboration.

We conclude by discussing very briefly some extensions of our setup:

Non-tracker learning about the number of trackers. A reasonable assumption is that without market entry non-trackers also learn in the long run whether they face a non-tracker or a tracker. If the rival is known to be a tracker with probability one, non-trackers set the monopoly price to target captives. If a non-tracker knows the rival is another non-tracker, its prices follow the usual Varian (1980) model.

Tracker learning about the number of non-trackers. If a tracker does not know whether it is competing against a non-tracker or a tracker, tracker-optimal play in continuous pricing for low enough τ involves a learning phase where the rival type is revealed. The first periods are hence invested in undercutting rivals and observing the reactions: a non-tracker cannot retaliate. Thereafter, revealed trackers collude.

Conversely, if the expected number of non-trackers remains low, a collusive equilibrium where non-trackers collude with trackers exists. In this case, the expected gain from exposing a non-tracker is low but the expected cost of tracker retaliation significant.

Endogeneous trackers and shoppers in a search market. The number of trackers and shoppes can be endogenized, for example, by assuming that the (menu) cost c of tracking market prices follows a distribution H and the (search) cost s of observing the lowest market price the distribution W, both supported over the unit interval. The firms and consumers with the lowest cost realizations thus become trackers and shoppers, respectively. The equilibrium measure of trackers $\tau = H(c) \in (0, 1)$ is defined by the fixed-point condition

$$\pi_0(\mu, \tau) = \pi_1(\mu, \tau) - c,$$

and that of shoppers $\mu = W(s) \in (0, 1)$ by the jointly determined equilibrium condition

$$CS_0(\mu, \tau) = CS_1(\mu, \tau) - s.$$

For example, cost distributions with $H(0.5) = 0.25$ and $W(0.5) = 0.1$ would allow to sustain a duopoly equilibrium where approximately one half of the firms become trackers

[39]We hope our work inspires future theorists to, e.g., fully endogenize the timing of price commitments, which would be an important extension but outside the scope of the paper; see Caruana and Einav (2008) for the problems this extension would involve.

and one half of the consumers are shoppers in expectation (Figs. 4.b and 5.b).

Endogeneous trackers with a market-making intermediary. A market-making intermediary can take advantage of this feature by providing a tracking technology to firms for a fee f. Because the intermediary profits from payoff differences, $f = \Delta\pi := \pi_1 - \pi_0$, its optimal policy trades off the number of trackers $\tau = H(f)$, who pay the intermediary, and the size of the fee, f.

$$\max_f H(f)\Delta\pi(f).$$

The problem has either an interior solution where some firms remain non-trackers, defined by $\Delta\pi(f)(1+\frac{\partial\Delta\pi(f)}{\partial f}) = 0$, or a corner solution $\tau = 1$ with only trackers. Notably, because the payoff difference $\Delta\pi(f)$ is positive in the neighborhood of $\tau = 1$, a firm remains willing to pay for tracking even when all other firms are trackers. This kind of intermediation reduces consumer surplus.[40]

Regulating the frequency of price change. As humble policy advice, our research suggests that regulators should perhaps tap more vigilantly into the potential for regulating the frequency of online price changes. Price dispersion has no intrinsic value in the kind of markets we analyze but, as long as algorithms discount future payoffs, delaying a punishment weakens the punishment – disturbing thus collusion. We find that firms price in mixed strategies if they are uncertain about (i) rival's price commitments or (ii) their own ability to respond. This leaves more surplus to consumers.

Efficient regulation would require trading off consumer gains with the ability to respond to market shocks and the inflation costs of sticky prices (Miklós-Thal and Tucker, 2019; Wang and Werning, 2022). Extrapolating on Robert and Stahl (1993), Ater and Rigbi (2023) also warn that the frequency of price change is interlinked with consumer search and advertising. Alternative recently proposed solutions to algorithmic collusion include, e.g., intervening in the algorithm learning process (Abada and Lambin, 2023; Asker et al., 2021). Evaluating the tradeoffs is left for future studies.

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[40] Johnson et al. (2020) show that unsupervised pricing algorithms in a platform environment set either constant or cyclical prices that both split profits nearly equally among firms. Price-based prominence order of firms that puts weight on historical prices helps to discipline price setting at the platform.

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Appendix

Proof of Lemma 1

The proof is by induction in t.

Initial step. We show that the claim holds for $t = \frac{n-1}{n}$ and $t = \frac{n-2}{n}$.

Part I. $t = \frac{n-1}{n}$. Because the game ends after a single period, a firm's payoff is summarized as follows

$$\begin{cases} \frac{1}{n}\frac{1-\mu}{2}p^i \text{ for } p^i > p^j \\ \frac{1}{n}\frac{1}{2}p^i \text{ for } p^i = p^j \\ \frac{1}{n}\frac{1+\mu}{2}p^i \text{ for } p^i < p^j \end{cases}$$

The claim of Lemma 1 therefore holds trivially as

$$\frac{1}{n}\frac{1+\mu}{2}p^i.$$

are increasing in p^i and

$$\frac{1}{n}\frac{1-\mu}{2}1$$

exceeds

$$\frac{1}{n}\frac{1+\mu}{2}(p^j-\epsilon)$$

if and only if $p^j - \epsilon < \underline{p}$.

Part II. $t = \frac{n-2}{n}$. Now, prices remain constant for two consecutive periods before an opportunity to change prices arrives. Supposing the rival employs the price cutoff $\underline{p}$, a firm has now three reasonable price strategies: $p^i = p_j - \epsilon$ (highest price below rival), $p^i = \underline{p}$ (highest price a rival will not undercut) and $p^i = 1$ (highest price above rival).

Case 1. $\underline{p} < p^j - \epsilon$ and $p^j < 1$.

$$\begin{aligned}
\pi(p^j-\epsilon|p^j) &= \frac{1}{n}\frac{1+\mu}{2}(p^j-\epsilon) + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1-\mu}{2}(p^j-\epsilon),\\
\pi(\underline{p}|p^j) &= \frac{1}{n}\frac{1+\mu}{2}\underline{p} + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1+\mu}{2}\underline{p} =\\
\pi(1|p^j) &= \frac{1}{n}\frac{1-\mu}{2}1 + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1-\mu}{2}1,
\end{aligned}$$

where $\pi(\underline{p}|p^j) < \pi(p^j-\epsilon|p^j)$ iff $\underline{p}(1+\mu) = 1-\mu < p^j - \epsilon$, and $\pi(1|p^j) < \pi(p^j-\epsilon|p^j)$ iff $1-\mu < p^j - \epsilon$.

Note that, if tracker j's price cutoff increases to $\hat{p} > \underline{p}$, then tracker i's price cutoff decreases to $\hat{p} < \underline{p}$ because $\pi(\underline{p}|p^j) > \pi(1|p^j)$ under a higher rival cutoff. By the symmetry assumption, it is therefore necessary that $\hat{p} = \underline{p}$ to make $BR^i(\hat{p}) = BR^j(\hat{p}) = 1$ hold for each firm.

Case 2. $p^j = 1$.

$$\begin{aligned}
\pi(1-\epsilon|1) &= \frac{1}{n}\frac{1+\mu}{2}(1-\epsilon) + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1-\mu}{2}(1-\epsilon) >\\
\pi(1|1) &= \frac{1}{n}\frac{1}{2}1 + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1-\mu}{2}1 >\\
\pi(\underline{p}|1) &= \frac{1}{n}\frac{1+\mu}{2}\underline{p} + \frac{1}{n}\left(1-\frac{1}{n}\right)\frac{1+\mu}{2}\underline{p}.
\end{aligned}$$

Case 3. if $\underline{p} \geq p^j - \epsilon$.

$$\pi(p^j - \epsilon|p^j) = \frac{1}{n}\frac{1+\mu}{2}(p^j - \epsilon) + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1+\mu}{2}(p^j - \epsilon) <$$
$$\pi(1|p^j) = \frac{1}{n}\frac{1-\mu}{2}1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1-\mu}{2}1.$$

Induction step. We show that, if the claim holds for $t > \frac{2}{n}$, the claim also holds for $t - 1 > \frac{1}{n}$.

Case 1. $\underline{p} < p^j - \epsilon$ and $p^j < 1$. We now take into account the rival's assumed best response to a price.

$$\pi(p^j - \epsilon|p^j) + V_t^i(p^j - 2\epsilon) = \frac{1}{n}\frac{1+\mu}{2}(p^j - \epsilon) + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1-\mu}{2}(p^j - \epsilon) + V_t^i(p^j - 2\epsilon),$$
$$\pi(\underline{p}|p^j) + V_t^i(1) = \frac{1}{n}\frac{1+\mu}{2}\underline{p} + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1+\mu}{2}\underline{p} + V_t^i(1),$$
$$\pi(1|p^j) + V_t^i(1-\epsilon) = \frac{1}{n}\frac{1-\mu}{2}1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1-\mu}{2}1 + V_t^i(1-\epsilon),$$

where again $\pi(\underline{p}|p^j) < \pi(p^j - \epsilon|p^j)$ iff $\underline{p}(1+\mu) = 1 - \mu < p^j - \epsilon$, and $\pi(1|p^j) < \pi(p^j - \epsilon|p^j)$ iff $1 - \mu < p^j - \epsilon$.

The claim thus holds at $t - 1$ for small ϵ, and large n because

$$V_t^i(1-\mu) \to V_t^i(1) \to V_t^i(1-\epsilon) \text{ as } \epsilon \to 0, n \to \infty.$$

The remaining Cases 2. and 3. are simple and left to the reader. □

Proof of Lemma 2

The claim follows from Lemma 1 Part II. Case 1. and the monotonicity of payoffs in rival price $\pi(p^j - \epsilon|p^j) < \pi(q^j - \epsilon|q^j)$ iff $p^j < q^j$. □

Proof of Proposition 6

By Lemma 2, the maximum expected profit is attained by tracker 1 choosing $p^1 = 1$ at $t = \frac{1}{n}$. As $n\epsilon \to 0$, all prices in the cycle $1, 1-\epsilon, 1-2\epsilon, ..., 1-(n-1)\epsilon$ are higher than $1 - n\epsilon \to 1$. The expected firm profit thus exceeds

$$\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}\right)(1 - n\epsilon) \to \left(\frac{1-\mu}{2} + \frac{1+\mu}{2}\right) = \frac{1}{2} \quad \square$$

Proof of Proposition 7

As $n\epsilon \to \infty$, all prices in the cycle $1, 1-\epsilon, 1-2\epsilon, ..., 1-\mu, \underline{p}$ occur infinitely often. One cycle takes $\frac{\mu}{\epsilon}+1$ periods to go through.

The relative frequency of all prices within $[1-\mu, 1]$ is thus $\frac{\frac{\mu}{\epsilon}}{\frac{\mu}{\epsilon}+1} \to 1$ for $\epsilon \to 0$ while that of price $\underline{p}$ is $\frac{1}{\frac{\mu}{\epsilon}+1} \to 0$ for $\epsilon \to 0$, giving

$$\frac{1+(1-\mu)}{2} = \frac{2-\mu}{2},$$

as the mean price over a cycle. $\square$

Proof of Lemma 4

Case 1. If a non-tracker believes that even two trackers will undercut its price, it is willing to offer as low a price as p_2''. Two trackers are in turn willing to undercut this price as long as $\underline{p}_2 < p_2''$.

$$\underline{p}_2 < p_2''$$
$$\frac{1-\mu}{\frac{3}{2}\mu+1-\mu} < \frac{1-\mu}{3(1-\tau)^2\mu+1-\mu}$$
$$\tau > 1-\frac{1}{\sqrt{2}}$$

Case 2. If a non-tracker believes that a single tracker will undercut its price but two trackers won't, it is willing to offer as low a price as p_1''. A single tracker is willing to undercut this price but two trackers not if $\underline{p}_1 < p_1'' \leq \underline{p}_2$.

$$\underline{p}_1 < p_1'' \leq \underline{p}_2$$
$$\frac{1-\mu}{\frac{3}{1}\mu+1-\mu} < \frac{1-\mu}{3(1-\tau)^2\mu+3\tau^2\mu+1-\mu} \leq \frac{1-\mu}{\frac{3}{2}\mu+1-\mu}$$
$$1 > (1-\tau)^2+\tau^2 \geq \frac{1}{2}$$

Case 3. If a non-tracker believes that a tracker does not undercut its price, it is willing to offer as low a price as p_0''. A single tracker is not willing to undercut this price by any amount ϵ if $p_0'' \leq \underline{p}_1$.

$$p_0'' \leq \underline{p}_1$$
$$\frac{1-\mu}{3\mu+1-\mu} \leq \frac{1-\mu}{\frac{3}{1}\mu+1-\mu}$$
$$1 \geq 1 \quad \square$$

This shows that $p_0'' = p'' \leq \underline{p}_1$ cannot be satisfied as an inequality $p_0'' = p'' < \underline{p}_1$. It is thus impossible to sustain an equilibrium with two gaps, where no tracker undercuts $[p_0'', \underline{p}_1]$, a single tracker undercuts $[\underline{p}_1 + a_1, \underline{p}_2]$, and two trackers undercut $[\underline{p}_2 + a_2, 1]$. Limit pricing is also impossible because commitment to p_0'' gives a non-tracker lower payoff than monopoly pricing when other non-trackers also commit to the same price.

Proof of Proposition 3

By Lemma 4, $\underline{p}_1 < \underline{p}_2 < p_2''$ is satisfied for $\tau > 1 - \frac{1}{\sqrt{2}}$. Thus, an equilibrium where non-trackers mix prices across $[p_2'', 1]$ and trackers collude to $\tilde{p} - \epsilon > \underline{p}_2 > \underline{p}_1$ exists. The equilibrium price distribution of a non-tracker is given by the requirement that non-tracker profits are the same for all prices in $[p_2'', 1]$.

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu(1-\tau)^2(1-F(p^i))^2\right)p^i$$
$$F(p^i) = 1 - \sqrt{\frac{1-\mu}{3\mu(1-\tau)^2}\frac{1-p^i}{p^i}}$$

Proof of Proposition 4

By Lemma 4, $\underline{p}_1 < p_1'' \leq \underline{p}_2$ is satiefied for $\tau > 0$. The gap size a_2 is defined in the main text.

The equilibrium price distribution of a non-tracker for $[\underline{p}_2 + a_2, 1]$ is given by

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu(1-\tau)^2(1-F(p^i))^2\right)p^i$$
$$F(p^i) = 1 - \sqrt{\frac{1-\mu}{3\mu(1-\tau)^2}\frac{1-p^i}{p^i}}$$

The equilibrium price distribution of a non-tracker for $[p_1'', \underline{p}_2]$ is given by

$$\frac{1-\mu}{3} = \left(\frac{1-\mu}{3} + \mu((1-\tau)^2 + \tau^2)(1 - F(p^i))^2 \right) p^i$$
$$F(p^i) = 1 - \sqrt{\frac{1-\mu}{3\mu((1-\tau)^2 + \tau^2)} \frac{1-p^i}{p^i}}$$

Punishment price cycle in a triopoly

the profit from a marginal price discount $\epsilon < p^j - \underline{p}$ is

$$\frac{1}{n}\frac{1+2\mu}{3}(p^j - \epsilon) + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1-\mu}{3}(p^j - \epsilon) + \frac{1}{n}\left(1 - \frac{1}{n}\right)^2 \frac{1-\mu}{3}(p^j - \epsilon)$$

because the rivals offer a smaller price in the coming two periods. However, the profit from a large price discount $p^j - \underline{p}$ is

$$\frac{1}{n}\frac{1+2\mu}{3}\underline{p} + \frac{1}{n}\left(1 - \frac{1}{n}\right)\frac{1+2\mu}{3}\underline{p} + \frac{1}{n}\left(1 - \frac{1}{n}\right)^2 \frac{1+2\mu}{3}\underline{p}$$

since this price will not be undercut in the next three periods.

The low price bound in this case is $\underline{p} = \frac{1-\mu}{1+2\mu}$. Thus, the price p^j at which a tracker rather offers a large discount $p^j - \underline{p}$ than a small discount ϵ, as $\nu \to \infty$ and $\epsilon \to 0$, is given by

$$\frac{3(1+2\mu)}{3+2\mu-2\mu}\underline{p} = 1 - \mu.$$

Proof of Lemma 5

Denote by $p''(\bar{p})$ the lower bound on non-tracker prices when the upper bound is $\bar{p}$. Accounting for the fact that $p''(\underline{p}_2) < p''(1)$, the proof is the same as for Lemma 4, the focus now being on Case 2. and Case 3.

The profit of a non-tracker is

$$\pi(\underline{p}_2) := \left(\frac{1-\mu}{3} + \mu\tau^2 \right) \underline{p}_2$$
$$\pi(p_1''(\underline{p}_2)) := \left(\frac{1-\mu}{3} + \mu\tau^2 + \mu(1-\tau)^2 \right) p_1''(\underline{p}_2)$$
$$\pi(p_0''(\underline{p}_2)) := \left(\frac{1-\mu}{3} + \mu \right) p_0''(\underline{p}_2)$$

The lowest price a non-tracker is willing to offer is thus

$$p_1''(\underline{p}_2) = \frac{3\tau^2\mu + 1 - \mu}{3(1-\tau)^2\mu + 3\tau^2\mu + 1 - \mu}\underline{p}_2$$

if the price is undercut by one tracker and

$$p_0''(\underline{p}_2) = \frac{3\tau^2\mu + 1 - \mu}{3\mu + 1 - \mu}\underline{p}_2$$

if the price not is undercut by a tracker.

Case 2. If a non-tracker believes that a single tracker will undercut its price but two trackers won't, it is willing to offer as low a price as $p_1''(\underline{p}_2) < \underline{p}_2$. A single tracker is willing to undercut this price but two trackers not if $\underline{p}_1 < p_1''(\underline{p}_2)$.

$$\frac{1-\mu}{\frac{3}{1}\mu + 1 - \mu} < \frac{3\tau^2\mu + 1 - \mu}{3(1-\tau)^2\mu + 3\tau^2\mu + 1 - \mu}\frac{1-\mu}{\frac{3}{2}\mu + 1 - \mu}$$
$$\frac{\frac{3}{2}\mu + 1 - \mu}{\frac{3}{1}\mu + 1 - \mu} < \frac{3\tau^2\mu + 1 - \mu}{3(1-\tau)^2\mu + 3\tau^2\mu + 1 - \mu},$$

which holds as long as $\tau^2 > \frac{1}{2}$.

Case 3. If a non-tracker believes that a tracker does not undercut its price, it is willing to offer as low a price as $p_0''(\underline{p}_2) < \underline{p}_2$. A single tracker is not willing to undercut this price by any amount ϵ if $p_0''(\underline{p}_2) \leq \underline{p}_1$.

$$\frac{1-\mu}{3\mu + 1 - \mu} \leq \frac{3\tau^2\mu + 1 - \mu}{3\mu + 1 - \mu}\frac{1-\mu}{\frac{3}{1}\mu + 1 - \mu}$$

This is obviously impossible. □

Proof of Proposition 5

By Lemma 5, $\tau \geq \frac{1}{\sqrt{2}}$ implies $p_0'' \leq \underline{p}_1$. □

Tracker and non-tracker profits

In the limit-price equilibrium Alternative III, the profit of a non-tracker is given by

$$\pi_0 = \left(\frac{1-\mu}{3} + \mu\tau^2\right)\frac{1-\mu}{\frac{3}{2}\mu + 1 - \mu}.$$

In the alternative price equilibria, the profit of a non-tracker is invariably $\pi_0 = \frac{1-\mu}{3}$.

Instead, tracker profit depends on both the pricing behavior of non-trackers and the number of trackers in the market.

Alternative I. In this case, trackers always undercut non-trackers by ϵ if a non-tracker is present in the market. The expected tracker profit is thus

$$\pi_1 = \tau^2 \frac{1}{3} + 2\tau(1-\tau)\frac{2+\mu}{3}\frac{E(p_0)}{2} + (1-\tau)^2 \frac{1+2\mu}{3} E(p_{(1)}),$$

where

$$E(p_0) = \sqrt{\frac{1-\mu}{3(1-\tau)^2\mu}} \tan^{-1}\left(\sqrt{\frac{1}{p_2''}-1}\right)$$
$$E(p_{(1)}) = -\frac{1-\mu}{3(1-\tau)^2\mu} \ln p_2''$$

Alternative II. In this case, two trackers only undercut non-trackers if their price exceeds $\underline{p}_2$, which allows us to express the expected tracker profit as

$$\pi_1 = \tau^2 \frac{1}{3} + 2\tau(1-\tau)\left(F(\underline{p}_2)\frac{1-\mu}{3} + (1-F(\underline{p}_2))\frac{2+\mu}{3}\frac{E(p_0|p_0 > \underline{p}_2)}{2}\right) + (1-\tau)^2 \frac{1+2\mu}{3} E(p_{(1)}),$$

where

$$F(\underline{p}_2) = 1 - \sqrt{\frac{\frac{1-\mu}{3\underline{p}_2} - \frac{1-\mu}{3} - \mu\tau^2}{\mu(1-\tau)^2}}$$
$$E(p_0|p_0 > \underline{p}_2 + a_2) = \sqrt{\frac{1-\mu}{3(1-\tau)^2\mu}} \tan^{-1}\left(\sqrt{\frac{1}{\underline{p}_2 + a_2} - 1}\right)\frac{1}{1-F(\underline{p}_2)}$$
$$E(p_{(1)}) = -\frac{1-\mu}{3(1-\tau)^2\mu}\ln(\underline{p}_2 + a_2) + \frac{1-\mu}{3(1-\tau)^2\mu}(\ln \underline{p}_2 - \ln p_1'')$$

Alternative III. In this case, two trackers never undercut non-trackers. The expected tracker profit is therefore

$$\pi_1 = \tau^2 \frac{1}{3} + 2\tau(1-\tau)\frac{1-\mu}{3} + (1-\tau)^2 \frac{1+2\mu}{3} E(p_{(1)}),$$

where

$$E(p_{(1)}) = \frac{\pi_0}{(1-\tau)^2\mu}(\ln \underline{p}_2 - \ln p_1''),$$

where π_0 denotes the profit of a non-tracker.

The total expected industry profit of trackers and non-trackers is given by

$$\Pi = \tau^3 1 + 3\tau^2(1-\tau)\left(\frac{2+\mu}{3}E(p_0|p_0 > \underline{p}_2)(1-F(\underline{p}_2)) + F(\underline{p}_2)\frac{2(1-\mu)}{3} + \pi_0\right) + \\ 3\tau(1-\tau)^2\left(\frac{1+2\mu}{3}E(p_{(1)}) + 2\pi_0\right) + (1-\tau)^3 3\pi_0$$

and the joint consumer surplus of shoppers and non-shoppers by $1-\Pi$.

Alternative I. Non-tracker equilibrium price distribution F is determined by

$$\pi_0 = \left(\frac{1-\mu}{3} + (1-\tau)^2\mu(1-F(p^i))^2\right)p^i$$

which gives

$$F(p^i) = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)^{1/2}$$

$$f(p^i) = 1/2\left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)^{-1/2}\frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2}$$

$$F_{(1|2)}(p^i) = 1-(1-F(p^i))^2 = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)$$

$$f_{(1|2)}(p^i) = \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2}$$

where $F_{(1|2)}$ denotes the distribution function of a minimum of two non-tracker prices and $f_{(1|2)}$ the associated density function.

The expected non-tracker price and the expected minimum of two prices are

$$\int_{p_2''}^1 f(p^i)p^i dp^i = \int_{p_2''}^1 1/2\left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)^{-1/2}\frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{p^i}dp^i$$

$$= \left(\frac{1-\mu}{3(1-\tau)^2\mu}\right)^{1/2}\int_{p_2''}^1 1/2\left(\frac{1}{p^i}-1\right)^{-1/2}\frac{1}{p^i}dp^i$$

$$= \left(\frac{1-\mu}{3(1-\tau)^2\mu}\right)^{1/2}\tan^{-1}\left(\frac{1}{p_2''}-1\right)^{1/2}$$

$$\int_{p_2''}^1 f_{(1|2)}(p^i)p^i dp^i = \int_{p_2''}^1 \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{p^i}dp^i = -\frac{1-\mu}{3(1-\tau)^2\mu}\ln p_2''$$

Alternative II. Non-tracker equilibrium price distribution F is determined piece-wise by

$$\pi_0 = \left(\frac{1-\mu}{3} + (1-\tau)^2\mu(1-F(p^i))^2\right)p^i \Longrightarrow F(p^i) = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)^{1/2}$$

for $p^i > \underline{p}_2 + a_2$ and by

$$\pi_0 = \left(\frac{1-\mu}{3} + \tau^2\mu + (1-\tau)^2\mu(1-F(p^i))^2\right)p^i \Longrightarrow F(p^i) = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3} - \tau^2\mu}{(1-\tau)^2\mu}\right)^{1/2}$$

for $p^i \leq \underline{p}_2$, which gives

$$f(p^i) = \begin{cases} 1/2\left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right)^{-1/2} \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2} & \text{for } p^i > \underline{p}_2 + a_2 \\ 1/2\left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3} - \tau^2\mu}{(1-\tau)^2\mu}\right)^{-1/2} \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2} & \text{for } p^i \leq \underline{p}_2 \end{cases}$$

$$F_{(1|2)}(p^i) = \begin{cases} 1 - (1-F(p^i))^2 = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}\right), & \text{for } p^i > \underline{p}_2 + a_2 \\ 1 - (1-F(p^i))^2 = 1 - \left(\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3} - \tau^2\mu}{(1-\tau)^2\mu}\right), & \text{for } p^i \leq \underline{p}_2 \end{cases}$$

$$f_{(1|2)}(p^i) = \begin{cases} \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2}, & \text{for } p^i > \underline{p}_2 + a_2 \\ \\ \frac{1-\mu}{3(1-\tau)^2\mu}\frac{1}{(p^i)^2}, & \text{for } p^i \leq \underline{p}_2 \end{cases}$$

Now, the expected non-tracker price is

$$\int_{\underline{p}_2+a_2}^{1} f(p^i)p^i dp^i + \int_{p_1''}^{\underline{p}_2} f(p^i)p^i dp^i =$$

$$\frac{1-\mu}{3(1-\tau)^2\mu}\left(\int_{\underline{p}_2+a_2}^{1} \frac{1}{2\sqrt{\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3}}{(1-\tau)^2\mu}}}\frac{1}{p^i}dp^i + \int_{p_1''}^{\underline{p}_2} \frac{1}{2\sqrt{\frac{\frac{1-\mu}{3p^i} - \frac{1-\mu}{3} - \mu\tau^2}{(1-\tau)^2\mu}}}\frac{1}{p^i}dp^i\right)$$

$$= \left(\frac{1-\mu}{3(1-\tau)^2\mu}\right)^{1/2} \tan^{-1}\left(\frac{1}{\underline{p}_2 + a_2} - 1\right)^{1/2} +$$

$$-\frac{\frac{1-\mu}{3(1-\tau)^2\mu}}{\sqrt{\frac{1+2\mu}{3(1-\tau)^2\mu}}}\tan^{-1}\left(\frac{1-\mu}{1+2\mu}\frac{1}{\underline{p}_2} - 1\right)^{1/2} + \frac{\frac{1-\mu}{3(1-\tau)^2\mu}}{\sqrt{\frac{1+2\mu}{3(1-\tau)^2\mu}}}\tan^{-1}\left(\frac{1-\mu}{1+2\mu}\frac{1}{p_1''} - 1\right)^{1/2}$$

and the expected minimum of two prices is

$$\int_{\underline{p}_2+a_2}^{1} f_{(1|2)}(p^i)p^i dp^i + \int_{p_1''}^{\underline{p}_2} f_{(1|2)}(p^i)p^i dp^i = -\frac{1-\mu}{3(1-\tau)^2\mu}\ln(\underline{p}_2 + a_2) + \frac{1-\mu}{3(1-\tau)^2\mu}(\ln \underline{p}_2 - \ln p_1'')$$

Discussion about the limit properties of ϵ and n

Minimum recognized discount. The limit properties of the minimum recognized discount $\epsilon > 0$ are of potential importance. This is the smallest discount size by which a firm is required to undercut its rival's price to capture all shoppers with certainty. The standard implicit assumption for continuous price strategies is that *any* tiny discount suffices to capture all shoppers. This assumption is innocuous in static models, such as Burdett and Judd (1983). However, the assumption may become dubious in dynamic markets if discounts accumulate in repeated application. Specifying the limit properties of ϵ and n is therefore important for understanding the nature of equilibria – Propositions 6 and 7.

Proposition 6 *Assume that the minimum recognized discount $\epsilon \to 0$ satisfies $n\epsilon \to 0$ as $n \to \infty$. In a discrete time price game, the unique equilibrium price sequence equals $(1, 1-\epsilon, 1-2\epsilon, ..., 1-n\epsilon)$. As $n \to \infty$ and $\epsilon \to 0$, the expected tracker profit is $\frac{1}{2}$ and the expected consumer surplus 0.*

If only a tiny discount $\epsilon < O(\frac{1}{n})$ suffices to capture any shopper, optimal prices remain close to unity during the entire time horizon. Because the consumer is equally likely to arrive in odd and even time periods, both firms have the same probability of winning any shoppers in this case. In the limit, each tracker thus obtains the profit of $\frac{1}{2}$, leaving almost no surplus to consumers, notwithstanding constant price cutting.

Proposition 7 *Assume that the minimum recognized discount $\epsilon \to 0$ satisfies $n\epsilon \to \infty$ as $n \to \infty$. In a discrete time price game, the unique equilibrium price sequence is the price cycle $\left(1, 1-\epsilon, 1-2\epsilon, ..., 1-\mu, \underline{p}\right)$. As $n \to \infty$ and $\epsilon \to 0$, the expected tracker profit is $\frac{1}{2}\frac{2-\mu}{2}$ and the expected consumer surplus $1 - \frac{2-\mu}{2}$.*

If only discounts exceeding a minimum discount size $\epsilon > O(\frac{1}{n})$ are recognized by consumers, a price cycle perpetuates. Prices thus repeat the cycle shown in Figure 2 infinitely.[41] As prices only briefly revisit $\underline{p}$ but remain mostly within the set $[1-\mu, 1]$ during a price cycle, the average market price in this case where discounts accumulate is $\frac{2-\mu}{2}$ in the limit where prices repeat the cycle an infinite number of times.[42]

We proceed to consider the continuous time pricing game. As in the previous discrete time game, we find that the nature of equilibrium depends on the limit properties of the minimum deviation ϵ.

Lemma 6 (Folk theorem I) *Assume that the minimum recognized discount $\epsilon \to 0$ satisfies $n\epsilon \to 0$ as $n \to \infty$. In a continuous time price game, any price sequence where firms' continuation payoffs exceed $(1-t)\frac{1-\mu}{2}$ at t can be implemented in equilibrium.*

When the minimum recognized discount is vanishingly small, the start price of the

[41] More precisely speaking, if μ and $1-\underline{p}$ are not divisible by ϵ, the final values over cycle are $1-k\epsilon$ and $1-l\epsilon$ where $1-k\epsilon > \mu > 1-(k-1)\epsilon$ and $1-l\epsilon > 1-\underline{p} > 1-(l-1)\epsilon$.

[42] The remaining cases where $n\epsilon \to a \in (0, \infty)$ are non-generic and thus omitted.

punishment cycle matters. By starting the cycle at price $p > 1 - \mu$ at time t, firms acquire the payoffs $(1-t)\frac{p}{2}$. Firms keep undercutting each other over the cycle but the price p barely budges over $[t, 1]$.

Lemma 7 (Folk theorem II) *Assume that the minimum recognized discount $\epsilon \to 0$ satisfies $n\epsilon \to \infty$ as $n \to \infty$. In a continuous time price game, any price sequence where firms' continuation payoffs exceed $(1-t)\frac{2-\mu}{2}$ at t can be implemented in equilibrium.*

When the minimum recognized discount is distinctly larger, the start price of the punishment cycle is irrelevant. By starting the cycle at any price p at time t, firms acquire the payoffs $(1-t)\frac{2-\mu}{2}$. Firms rotate though the cycle infinitely many times over $[t, 1]$, the mean price being $\frac{2-\mu}{2}$.

Proposition 8 *In a continuous time price game, maximum tracker payoffs $\frac{1}{2}$ obtain in a collusive equilibrium where firms set price one under the threat of reverting to a punishment cycle started at a lower price $p \in (1-\mu, 1)$. The consumer obtains no expected surplus.*