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Tapio Palokangas*

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Abstract

Heterogeneous monopolists produce goods using either brown technology, which relies on labor and carbon energy, or green technology, which relies solely on labor. R&D firms enhance productivity using labor to outcompete existing monopolists, thereby driving economic growth. The extraction of carbon energy releases pollutants that harm production and increase the risk of environmental disaster. The government can optimally mitigate the distortions caused by pollution by a two-part Pigouvian tax on carbon energy, with one part being precautionary, applied only before any disaster occurs. When this tax is optimally set, R&D should neither be taxed nor subsidized.

Journal of Economic Literature: $H21$; $O32$; $O44$; $Q52$; $Q54$; $Q58$

Keywords: Emissions; pollution; R&D; endogenous growth; environmental disaster; precautionary policy

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1 Introduction

Emissions contribute to pollution, increasing the risk of a welfare-damaging environmental disaster. OECD countries counter this risk by taxing polluting inputs and subsidizing R&D for cleaner technologies. However, there has been little comprehensive analysis of optimal policy in an endogenously growing market economy where pollution heightens disaster risk. This study aims to fill that gap. The literature often assumes that the economic impact of pollution is immediate.[1] Since environmental disasters driven by pollution are expected to affect the economy mainly in the future, it is crucial to examine precautionary policies that mitigate the disaster risk beforehand.[2]

In this document, following Tsur and Zemel (2008, 2009), Polasky et al. (2011), and de Zeeuw and Zemel (2012), environmental degradation is treated as a low-frequency shock: a random regime shift that occurs only once, with the post-event regime persisting indefinitely. Recurrent events, where multiple shifts occur at random intervals, can be analyzed using the same methodology.[3] Many pollution models assume a smooth damage function, allowing policymakers to respond immediately to marginal pollution increases.[4] In such cases, precautionary policy is unnecessary. However, in this model, the damage function is discrete due to the regime shift.

Haurie and Moresino (2006) investigate the social planner's precautionary policy aimed at mitigating a potential catastrophe in an economy characterized by two distinct types of capital stocks: general productive physical capital and equipment specifically designed to reduce the social costs of the anticipated catastrophe. In this context, it is optimal to maintain a substantial stock of the alleviating capital. Other authors have examined the concept of the green paradox:[5] if investment in physical capital is irreversible across several sectors and the implementation of corrective taxes is delayed, the

[1]Most authors use this assumption to simplify mathematical models. Acemoglu et al. (2016) also cite the ability to calibrate model parameters using US microeconomic data.

[2]Immediate effects can be included in the model by incorporating pollution into utility or production functions. However, this study focuses on precautionary policy and ignores immediate effects to avoid unnecessary technical complications.

[3]Cf., de Zeeuw and Zemel (2012).

[4]E.g., van der Ploeg and de Zeeuw (1992), and Dockner and Long (1993).

[5]Cf., Sinn (2008), Valente (2011), Tsur and Zemel (2011), Smulders et al. (2012), and Afonso et al. (2021).

pre-implementation responses of the emitting sectors may counteract some of their post-implementation adjustments. To avoid excessive complexity in the model presented in this document, capital stocks are excluded, as incorporating physical capital, endogenous technological change, and an endogenous random shock into the same model would be too intricate.[6]

Tsur and Zemel (2008, 2009) and de Zeeuw and Zemel (2012) examine an economy where firms use two energy inputs that are perfect substitutes: a green input that does not emit pollutants, and a brown input whose emissions accumulate in a "hazardous" stock, posing a risk of triggering damaging changes. They demonstrate that optimal policy necessitates a Pigouvian tax on the "hazardous" input. In the model presented in this document, firms can switch between the green sector, which employs only labor, and the brown sector, which uses both labor and emitting inputs. Consequently, a single Pigouvian tax is insufficient for achieving optimal policy.

Grimaud and Tournemaine (2007) develop a model in which firms conduct in-house R&D using educated labor to reduce pollution emissions, while education directly contributes to welfare. Consequently, education enhances both individual productivity and utility, whereas pollution diminishes welfare. Additionally, a pollution tax increases the cost of polluting goods, reallocating resources towards education and fostering economic growth. In the model presented in this document, education is considered solely as a transformation of labor into R&D inputs, with no direct impact on welfare.

In Silva et al. (2013), both horizontal innovation and pollution-reducing knowledge are driven by endogenous technical change. This scenario creates two distortions, which are addressed through an emissions tax and a consumption subsidy. When pollution-reducing knowledge becomes sufficiently efficient to surpass horizontal innovation, it is possible to achieve lower emissions alongside higher output levels and growth rates. In the model presented in this document, there is a fixed number of firms that can switch between two sectors with distinct productivity-enhancing innovation processes: the brown sector, which emits carbon fumes, and the green sector, which does not. Consequently, there is no horizontal innovation.

[6] Golosov et al. (2014) demonstrate that, despite some complications, physical capital can be integrated as an additional production factor in an endogenous growth model with minimal impact on the results.

Palokangas (2021) investigates optimal taxation in an economy where families decide on fertility rates and invest in capital and health care that reduces mortality. In this context, capital accumulation and population growth pose a risk of triggering an environmental shock, leading to a sudden increase in mortality. This must be addressed by precautionary taxes on both capital income and health care. In the model presented in this document, the pollution externality is similar to that in Palokangas (2021). However, instead of population growth, the focus is on R&D that drives technological change.

The rest of this document is structured as follows: Section 2 models the economy's technology. Section 3 illustrates the stationary-state equilibrium, where households save, monopolists produce goods, and R&D firms strive for higher efficiency to outcompete them. In this context, Section 4 examines optimal policies, and Section 5 concludes with a summary of the findings.

2 Technology

2.1 Utility

In the model, time t is continuous. The representative household derives utility from its consumption c, and the observed *state of nature*, q, as follows:[7]

$$u(c,q) \doteq q\frac{c^{1-\sigma}}{1-\sigma}, \quad u_{cc} \doteq \frac{\partial^2 u}{\partial^2 c} < 0, \quad \frac{cu_c}{u} = 1-\sigma \in (0,1), \tag{1}$$

where $\nu \geq 0$ is a constant and $\sigma \in (0,1) \cup (1,\infty)$ is the *constant rate of relative risk aversion*.[8]

2.2 Production

The consumption good is produced from a continuum of intermediate goods, $j \in [0,1]$, each of which can be produced using either brown or green technology. With *brown technology* $i = b$, output y_{bj} is produced from labor l_j

[7]The existence of a persistent stationary-state growth rate requires that the elasticity of utility u with respect to consumption c, $\frac{c}{u}\frac{\partial u}{\partial c}$, remains constant in the stationary state. Therefore, following Wälde (1999a, 1999b), the constant term $-\frac{1}{1-\sigma}$ is omitted from the standard utility function with the constant rate of relative risk aversion, $\frac{1}{1-\sigma}(c^{1-\sigma}-1)$.

[8]Given that some economists calibrate the effects of environmental stocks using models where the constant rate of relative risk aversion, σ, is greater than one, the model has been extended to accommodate this scenario as well.

and carbon energy m_j according to the neoclassical production function f with constant returns to scale:[9]

$$y_{bj} = a_b\gamma(j)f(l_j, m_j), \;\; f_l \doteq \frac{\partial f}{\partial l_j} > 0, \;\; f_m \doteq \frac{\partial f}{\partial m_j} > 0, \quad \gamma' < 0,$$
$$f(0, m_j) = f(l_j, 0) = 0, \;\; f \text{ concave and linearly homogeneous,} \tag{2}$$

where a_b is the *state-of-the-art efficiency* and $\mu(j)$ the exogenous *producer-specific efficiency* using brown technology. With *green technology* $i = g$, output y_{gj} is produced from labor z_j according to

$$y_{gj} = a_g\mu(j)z_j, \quad \mu(0) = 0, \quad \mu' > 0, \tag{3}$$

where a_{gj} is the *state-of-the-art efficiency* and $\mu(j)$ the exogenous producer-specific efficiency using green technology.[10]

The set of production lines j utilizing technology $i \in \{b, g\}$, denoted as I_i, is referred to as *sector* i. The relative sizes of these sectors are given by

$$J \doteq \int_{j\in I_b} dj, \quad 1 - J \doteq \int_{j\in I_g} dj. \tag{4}$$

The output of the consumption good, y, is produced from the intermediate goods $j \in [0, 1] = I_b \cup I_g$ according to the CES function:

$$y = \left(\int_{j\in I_b} y_{bj}^{1-1/\varepsilon} dj + \int_{j\in I_g} y_{gj}^{1-1/\varepsilon} dj\right)^{\varepsilon/(\varepsilon-1)}, \quad \varepsilon > 1, \tag{5}$$

where ε is the constant *elasticity of substitution* between any pair of the inputs. Following Acemoglu et al. (2016), the immediate effect of pollution is modeled such that the stock of pollution, P, adversely impacts the quality of consumption c:

$$c = P^{-\nu}y, \quad \nu \geq 0, \tag{6}$$

where ν is the constant elasticity of consumption with respect to pollution.

[9]The function (2) can also be specified as $a_b f\big(\gamma(j)l_j, m_j\big)$, without altering the results.

[10]Following Acemoglu and Zilibotti (2001), production lines $j \in [0, 1]$ are in (2) and (3) organized so that higher values of j indicate greater effectiveness of green technology ($\mu' > 0$) and lower effectiveness of brown technology ($\gamma' < 0$) in production.

2.3 Spillover of knowledge

In each production line $j \in [0,1] = I_b \cup I_i$, a unique R&D process enhances the efficiency of that line, a unique R&D process enhances the efficiency of that line, a_{ij}, over time. Due to knowledge spillovers within the same sector i, innovations in production line $j \in I_i$ also directly benefit other producers $k \in I_i \setminus j$ that use the same technology i. Following Young (1998), Aghion and Howitt (1998), and Howitt (1999), spillovers in a sector are assumed to depend on the knowledge of the most advanced firm. Hence, the state-of-the-art efficiency of technology $i \in \{b, g\}$ is defined as follows:

$$a_i = \max_{j \in I_i} a_{ij}. \tag{7}$$

Because there can be knowledge spillovers also between the sectors $i \in \{b, g\}$, it is assumed that both the state-of-efficiency of the same sector, a_i, and the state-of-the-art efficiency of the other sector, a_{-i}, enhance the efficiency of R&D in that sector i, Ω_i. This relationship is described by the function[11]

$$\Omega_i(a_i, a_{-i}), \quad \frac{\partial \Omega_i}{\partial a_i} > 0, \quad \frac{\partial \Omega_i}{\partial a_{-i}} > 0, \quad \Omega_i \text{ concave and linearly homogeneous.} \tag{8}$$

2.4 Technological change

Over a short time interval dt, an attempt to improve efficiency in production line $j \in I_i$ through R&D input s_{ij} succeeds with probability $\delta s_{ij} dt$ in creating an *innovation* that enhances the efficiency a_{ij} of producing good $j \in I_i$ beyond the level $\Omega_i(a_i, a_{-i})$, where $\delta > 0$ is a constant:

$$\frac{da_{ij}}{\Omega_i(a_i, a_{-i})} = \delta s_{ij} dt. \tag{9}$$

Conversely, over that interval dt, it fails with probability $1 - \delta s_{ij} dt$. Over a short time interval dt, the state-of-the-art efficiency (7) increases by the sum

[11] In the one-sector models proposed by Young (1998), Aghion and Howitt (1998), and Howitt (1999), equation (9) is expressed as $\frac{da_{ij}}{a_i} = \delta s_{ij} dt$. In the model presented in this document, the spillover of knowledge between sectors is essential to maintain a stable stationary state, ensuring that the two sectors do not grow at different rates indefinitely.

of the probabilities (9) for the goods $j \in I_i$ as follows:

$$\frac{da_i}{\Omega_i(a_i, a_{-i})} = \int_{j\in I_i} \frac{da_{ij}}{\Omega_i(a_i, a_{-i})} dj = \delta \int_{j\in I_i} s_{ij} dj\, dt. \tag{10}$$

According to (10), the state-of-the-art efficiencies evolve as follows:

$$\begin{aligned} \dot{a}_b &\doteq \frac{da_b}{dt} = \delta\Omega_b(a_b, a_g) \int_{j\in I_b} s_{bj} dj, \quad a_b(T) = a_b^T, \\ \dot{a}_g &\doteq \frac{da_g}{dt} = \delta\Omega_g(a_g, a_b) \int_{j\in I_g} s_{gj} dj, \quad a_i(T) = a_i^T, \end{aligned} \tag{11}$$

where a_i^T is the efficiency a_i at the initial time T.

2.5 The labor market

Carbon energy $\int_{j\in I_b} m_j dj$ is extracted from the nature using labor. This generates increasing and convex costs Δ in terms labor:

$$\Delta\left(\int_{j\in I_b} m_j dj\right)_, \quad \Delta' > 0, \quad \Delta'' > 0. \tag{12}$$

Labor can be transformed into R&D input s_{ij} to improve efficiency in production line $j \in I_i$ within sector $i \in \{b, g\}$, but with increasing costs. For analytical convenience, this relationship is expressed in a quadratic form:

$$\frac{s_{bj}^2}{2\alpha}, \ \ j \in I_g, \quad \frac{s_{gj}^2}{2\alpha}, \ \ j \in I_g, \quad \alpha > 0, \tag{13}$$

where α is a constant.

The households supply a fixed amount L of labor. This supply is in equilibrium equal to labor inputs in production, $\int_{j\in I_b} z_j dj + \int_{j\in I_g} l_j dj$ [cf., (2) and (3)], the extraction costs of carbon energy, (12), and the R&D costs (13):

$$L = \int_{j\in I_b} z_j dj + \int_{j\in I_g} l_j dj + \Delta\left(\int_{j\in I_b} m_j dj\right) + \int_{j\in I_b} \frac{s_{bj}^2}{2\alpha} dj + \int_{j\in I_g} \frac{s_{gj}^2}{2\alpha} dj. \tag{14}$$

3 The stationary-state equilibrium

The inputs to production and R&D are denoted as vectors $\mathbf{z} \doteq \{z_j | \, j \in I_g\}$, $\mathbf{l} \doteq \{l_j | \, j \in I_b\}$, $\mathbf{m} \doteq \{m_j | \, j \in I_b\}$ and $\mathbf{s} \doteq \big\{s_{ij} | \, j \in I_i, \, i \in \{b, g\}\big\}$. In the remainder of this section, the following equilibrium is proven to exist in the model presented in section 2:

Definition. The economy is in a *stationary-state equilibrium*, if the following properties are met:
(i) The inputs of labor and carbon energy $(\mathbf{l}, \mathbf{m}, \mathbf{z}, \mathbf{s})$ and the relative size of the brown sector, $J \in (0, 1)$, remain constant over time.
(ii) Aggregate production y and the state-of-the-art efficiencies a_b and a_g grow at the same constant rate over time.

3.1 Households

At the initial time $t = T$, the representative household maximizes its utility (1) over the foreseeable future $t \in [T, \infty)$,

$$\int_T^\infty u(c, q) e^{\rho(T-t)} dt. \tag{15}$$

by consumption c, given the interest rate r and the state of nature, q. By the standard analysis, consumption c evolves according to the *Euler equation*

$$G \doteq \frac{\dot{c}}{c} = \frac{r - \rho}{\sigma} \quad \Leftrightarrow \quad r = \rho + \sigma G, \tag{16}$$

where G is the growth rate of consumption c.

3.2 Competitive producers

Because there is no money in the model, labor is chosen as the *numeraire*, so that the wage for labor is normalized at unity. Let τ represent the tax on carbon energy costs. Competitive producers set the price of carbon energy, v, equal to the marginal extraction cost of carbon energy [cf., (12)]

$$v = (1 + \tau)\Delta'. \tag{17}$$

Let p represent the price for consumption c. When intermediate good j is produced using technology $i \in \{b, g\}$, competitive producers set the marginal product of input y_{ij}, $p\frac{\partial y}{\partial y_{ij}}$, equal to the price of that input, p_{ij} [cf., (5)]:

$$p_{ij} = p\frac{\partial y}{\partial y_{ij}} = p\left(\frac{y}{y_{ij}}\right)^{1/\varepsilon} \text{ with } \frac{y_{ij}}{p_{ij}}\frac{\partial p_{ij}}{\partial y_{ij}} = -\frac{1}{\varepsilon}, \quad j \in I_i, \quad i \in \{b, g\}. \tag{18}$$

3.3 Production lines $j \in [0,1]$

Each intermediate good $j \in [0,1]$ is produced by a single monopolist, labeled accordingly. For each production line $j \in [0,1]$, there is also a competitive R&D firm, also labeled accordingly. These agents make their decisions in the following strategic order:

(i) Monopolist j chooses its technology $i \in \{b, g\}$.

(ii) R&D firm j attempts to outcompete the incumbent monopolist j using labor s_{ij}. If successful, it becomes the new monopolist j.[12]

(iii) Monopolist j produces good j using labor and carbon energy (l_j, m_j) with brown technology, or just labor z_j with green technology.

In the next subsections, this extensive form game is solved in reverse order.

3.4 Production

Monopolist j either maximizes its *brown-technology operating profit* $\pi_{bj} = p_{bj}y_{bj} - l_j - vm_j$, where p_{bj} represents the price for its output y_{bj}, v the price for carbon energy m_j, by the inputs of labor l_j and carbon energy m_j subject to the demand curve (18) and the production function (2) in the brown sector $i = b$, or it maximizes its *green-technology operating profit* $\pi_{gj} = p_{gj}y_{gj} - z_j$, where p_{gj} represents the price for its output y_{gj}, by labor input z_j subject to the demand curve (18) and the production function (3) in the green sector $i = g$. Noting (17) and (18), the first-order conditions of monopolist j are

$$\frac{\partial \pi_{bj}}{\partial l_j} = 0 \;\Leftrightarrow\; 1 = \left(1 - \frac{1}{\varepsilon}\right) p \left(\frac{y}{y_{bj}}\right)^{1/\varepsilon} \frac{\partial y_{bj}}{\partial l_j} = \left(1 - \frac{1}{\varepsilon}\right) p_{bj} a_b \gamma(j) f_l(l_j, m_j)$$

$$\text{for } j \in I_b, \tag{19}$$

$$\frac{\partial \pi_{bj}}{\partial m_j} = 0 \;\Leftrightarrow\; v = \left(1 - \frac{1}{\varepsilon}\right) p \left(\frac{y}{y_{bj}}\right)^{1/\varepsilon} \frac{\partial y_{bj}}{\partial m_j} = \underbrace{\left(1 - \frac{1}{\varepsilon}\right) p_{bj} a_b \gamma(j)}_{=1/f_l(l_j, m_j)} f_m(l_j, m_j)$$

$$\Leftrightarrow\; \frac{f_m(l_j, m_j)}{f_l(l_j, m_j)} = v = (1+\tau)\Delta'\left(\int_{j \in I_b} m_j dj\right) \text{ for } j \in I_b, \tag{20}$$

[12]The model can also be extended to the scenario where there are several R&D firms, including the incumbent monopolist j itself, without altering the results.

$$\frac{\partial \pi_{gj}}{\partial z_j} = 0 \quad \Leftrightarrow \quad 1 = \left(1 - \frac{1}{\varepsilon}\right)\left(\frac{y}{y_{gj}}\right)^{1/\varepsilon} \frac{\partial y_{gj}}{\partial z_j} = \left(1 - \frac{1}{\varepsilon}\right)\mu(j) p_{gj} a_g$$
$$\text{for } \ j \in I_g. \tag{21}$$

Because the production functions (2) and (5) are concave and the cost function (12) is strictly convex, the equilibrium (19)-(21) is unique. Noting (19)-(21) and the linear homogeneity of the function (2), $f = f_l l_j + f_m m_j$, the sectorial operating profits of monopolist j are obtained as follows:

$$\pi_{bj}(j, a_b, y, p, v) \doteq \max_{l_j, m_j} [p y^{1/\varepsilon} y_{bj}^{1-1/\varepsilon} - l_j - v m_j] = p_{bj} y_{bj}/\varepsilon = p y^{1/\varepsilon} y_{bj}^{1-1/\varepsilon}/\varepsilon$$
$$\text{for } \ j \in I_b, \tag{22}$$
$$\pi_{gj}(j, a_g, y, p, v) \doteq \max_{z_j} [p y^{1/\varepsilon} y_{gj}^{1-1/\varepsilon} - z_j] = p_{gj} y_{gj}/\varepsilon = p y^{1/\varepsilon} y_{gj}^{1-1/\varepsilon}/\varepsilon$$
$$\text{for } \ j \in I_g. \tag{23}$$

3.5 Research and development

Over a short time interval dt, R&D firm j that works for production line $j \in I_i$ in sector i earns expected revenue $\delta \pi_{ij} s_{ij} dt$, where $\delta s_{ij} dt$ is the probability of an innovation [cf., (9)] and π_{ij} is its forthcoming operating profit as monopolist j following the innovation [cf., (23)]. Over the interval dt, it is subject to R&D cost $\frac{s_{ij}^2}{2\alpha} dt$ [cf., (13)]. Thus, over the short time interval dt, the *expected profit* of R&D firm $j \in I_i$ is

$$\Pi_{ij} dt \doteq \left(\delta \pi_{ij} s_{ij} - \frac{s_{ij}^2}{2\alpha}\right) dt, \quad j \in I_i, \quad i \in \{b, g\}. \tag{24}$$

R&D firm $j \in I_i$ maximizes its expected profit (24) by its R&D input s_{ij}, given its operating profit π_{ij}. This yields $\frac{\partial \Pi_{ij}}{\partial s_{ij}} = 0$, which implies that a fixed proportion $\alpha\delta$ of the operating profits π_{ij} are invested in R&D:

$$s_{ij} = \alpha \delta \pi_{ij}, \quad j \in I_i, \quad i \in \{b, g\}. \tag{25}$$

With strictly convex costs functions (13), the equilibrium (25) is unique.

Inserting (25) into (24) yields the R&D firms' expected profits as follows:

$$\Pi_{ij} = \frac{\alpha}{2} (\delta \pi_{ij})^2_, \quad j \in I_i, \quad i \in \{b, g\}. \tag{26}$$

Furthermore, by (22), (23) and (25) it holds true that

$$\frac{\int_{j\in I_b} s_{bj}dj}{\int_{j\in I_g} s_{gk}dk} = \frac{\int_{j\in I_b} \pi_{bj}dj}{\int_{j\in I_g} \pi_{gk}dk} = \frac{\int_{j\in I_b} y_{bj}^{1-1/\varepsilon}dj}{\int_{j\in I_g} y_{gk}^{1-1/\varepsilon}dk}. \tag{27}$$

3.6 Choice of technology

By (5) and (17)-(23), there exists a *threshold monopolist* $j = J$ that is indifferent between the brown and green technologies $i \in \{b, g\}$.[13] The threshold J divides the production lines into two sectors as follows:

$$I_b = [0, J], \quad I_g = [J, 1]. \tag{28}$$

Because monopolists $j \in I_b \cup I_g = [0, 1]$ choose their technology $i \in \{b, g\}$, given the operating profits π_{ij}, they behave jointly as if there were a representative monopolist maximizing their joint expected profit [cf., (26)]

$$\Pi \doteq \int_0^J \Pi_{bj}dj + \int_J^1 \Pi_{gj}dj \tag{29}$$

by the relative size of the brown sector, J, given the operating profits π_{ij}. By (22), (23), (25) and (26)], this maximization results in the first-order condition that the threshold monopolist's operative profit π_{iJ}, output y_{iJ}, R&D s_{iJ} and the production cost are identical for the sectors $i \in \{b, g\}$:

$$\begin{aligned} &0 = \frac{\partial \Pi}{\partial J} = \Pi_{bJ} - \Pi_{gJ} = \frac{\alpha}{2}(\delta\pi_{bJ})^2 - \frac{\alpha}{2}(\delta\pi_{gJ})^2 \quad \Leftrightarrow \\ &\pi_{gJ} = \pi_{bJ} \quad \Leftrightarrow \quad y_{gJ} = y_{bJ} \quad \Leftrightarrow \quad s_{gJ} = s_{bJ} \quad \Leftrightarrow \quad l_J + vm_J = z_J. \end{aligned} \tag{30}$$

Because, by (22) and (23), π_{bJ} decreases and π_{gJ} increases with the increase in $J, \frac{\partial \Pi}{\partial J}$ decreases with the increase in J and the equilibrium (30) is unique.

3.7 Summary

In the system (16), (17)-(26) and (30), there is a stationary state where

- the inputs $(\mathbf{z}, \mathbf{l}, \mathbf{m}, \mathbf{s})$ and the relative size of the brown sector, J, remain constant,

[13]The concept of the threshold monopolist is from Acemoglu and Zilibotti (2001).

- consumption c, aggregate output y, the sectorial outputs (y_b, y_g), the efficiencies (a_b, a_g) grow at the same rate G,
- the consumption price p and the prices (p_b, p_g) for the intermediate goods fall relative to the *numeraire* (= the ordinary workers' wage) at the rate G, and
- the profits $(\pi_b, \pi_g, \Pi_b, \Pi_g)$ and all nominal magnitudes remain constant.

A *green shift* can be defined as a decrease in the relative size of the brown sector, J. Consequently, a marginal green shift is represented as $dJ < 0$.

4 The government

The government, being the only entity large enough to internalize these externalities, imposes a tax on carbon energy costs. The revenue generated from this tax is distributed to households through non-distorting transfers.

In this section, externalities and the concept of an environmental shock are first introduced. Next, the government's first-best solution is derived. Finally, the optimal taxation is obtained by comparing that solution with private agents' equilibrium conditions, as presented in the previous section.

The *environmental disaster* causes an abrupt decline in the state of nature, q, from 1 to a constant $\varphi \in (0,1)$, negatively impacting welfare $u(c,q)$. This event reduces the marginal utility of consumption, $u_c = qc^{-\sigma}$ [cf., (1)], at a given consumption level, c. The *relative damage of the environmental shock* to the state of nature, q, can be expressed in terms of the utility function (1) as follows:

$$\frac{u - u|_{q=\varphi}}{u} = \frac{u(c,q) - u(c,\varphi)}{u(c,q)} = 1 - \frac{\varphi}{q} > 0. \tag{31}$$

Aggregate emissions $\int_{j\in I_b} m_j dj$ contribute to the stock of pollution, P, but the nature absorbs a constant proportion β of that stock:

$$\dot{P} \doteq \frac{dP}{dt} = \int_{j\in I_b} m_j dj - \beta P, \quad \beta > 0, \quad P(T) = P_T, \tag{32}$$

where P_T is the initial value of P at time T. An increase in pollution P increases the *probability* of that disaster, ϕ. By this, the state of the nature is determined as follows:

$$q = \begin{cases} \varphi \in (0,1) & \text{with probability } \phi(P) \in (0,1), \\ 1 & \text{with probalility } 1-\phi(P), \end{cases} \quad \phi' > 0. \tag{33}$$

To run optimal policy, the government needs natural scientists' estimates on the pollution absorption rate β, the relative environmental decline due to the disaster, φ, and the marginal effect of pollution on disaster risk, ϕ'.

By the production functions (2), (3), (5) and (6), consumption c is a function of inputs $(\mathbf{z}, \mathbf{l}, \mathbf{m})$, the size of the brown sector, J [cf., (4)], efficiency (a_b, a_g) and pollution P:

$$c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P) = P^{-\nu} y, \quad y \doteq \left(\int_{j \in I_b} y_{bj}^{1-1/\varepsilon} dj + \int_{j \in I_g} y_j^{1-1/\varepsilon} dj \right)^{\varepsilon/(\varepsilon-1)}. \tag{34}$$

The government maximizes the household's utility (15) by the inputs of labor and carbon energy, $(\mathbf{z}, \mathbf{l}, \mathbf{m})$, R&D $\mathbf{s}$ and the relative size of the brown sector, J, subject to consumption (34), the equilibrium condition of the labor market, (14), the evolution of the efficiencies of the sectors, (a_b, a_g), (11), the evolution of pollution P, (32), and the environmental shock (33). This yields the following first-best conditions (cf., the Appendix):

$$\lambda = \begin{cases} qc^{-\sigma} \frac{\partial y}{y_{bj}} a_b f_l(l_j, m_j) & \text{for } j \in [0, J], \\ qc^{-\sigma} \frac{\partial y}{y_{bj}} a_g & \text{for } j \in [J, 1], \end{cases} \tag{35}$$

$$\frac{q}{c^\sigma} \frac{\varepsilon y^{1/\varepsilon}}{\varepsilon - 1} \left(y_{bJ}^{1-1/\varepsilon} - y_{gJ}^{1-1/\varepsilon} \right) = \lambda \left(l_J + \frac{f_m(l_J, m_J)}{f_l(l_J, m_J)} m_J - z_J + \frac{s_{gJ}^2 - s_{bJ}^2}{2\alpha} \right), \tag{36}$$

$$\varsigma = \rho + (\sigma - 1)G + \phi(P)(1 - \varphi/q) = r - G + \phi(P)(1 - \varphi/q), \tag{37}$$

$$\frac{\int_0^J s_{bj} dj}{\int_J^1 s_{gk} dk} = \frac{\int_0^J y_{bj}^{1-1/\varepsilon} dj}{\int_J^1 y_{gk}^{1-1/\varepsilon} dk}, \tag{38}$$

$$\frac{f_m(l_j, m_j)}{f_l(l_j, m_j)} - \Delta' =$$
$$\frac{1}{\lambda} \left[\frac{\sigma - 1}{c} \frac{\varepsilon y^{1/\varepsilon}}{\varepsilon - 1} \left(y_{bJ}^{1-1/\varepsilon} - y_{gJ}^{1-1/\varepsilon} \right) \frac{\beta}{m_J} + \frac{1}{\varsigma} \left(1 - \frac{\varphi}{q} \right) \phi' + (1 - \sigma) \frac{\nu}{P} \right], \quad j \in [0, J]. \tag{39}$$

Equations (35) establish a uniform shadow price of labor, λ, for the entire economy. The outputs y_{ij} of intermediate goods $i \in I_i$ in both sectors

$i \in \{b, g\}$ are adjusted to uphold this uniformity. By the private agents' equilibrium conditions (20), (25) and (30), the first-best condition (36) holds true. This result can be rephrased as follows:

Proposition 1 *The markets determine the relative size of the brown sector, J, optimally.*

The difference between the interest rate r and the growth rate G is termed the *effective rate of time preference*. Equation (37) defines the *expected effective rate of time preference*:

$$\varsigma \doteq \rho + (\sigma - 1)G + \phi(P)(1 - \varphi/q) = r - G + \phi(P)(1 - \varphi/q). \tag{40}$$

This is the sum of the effective rate of time preference, $r - G$, and the expected relative loss in utility, $\phi(P)(1 - \frac{\varphi}{q})$, where $\phi(P)$ is the probability of the disaster [cf., (33)] and $1 - \frac{\varphi}{q}$ represents the expected relative loss in welfare at the time of the disaster [cf., (1)]. Consistently, term $\left(1 - \frac{\varphi}{q}\right)\phi'$ represents the expected marginal relative disutility of pollution through the environmental disaster.

Condition (38) is the same as the private agents' equilibrium condition for R&D, (27). This result can be rephrased as follows:

Proposition 2 *R&D for any technology $i \in \{b, g\}$ should neither be taxed nor subsidized.*

Inserting (20) and (30) into condition (39) yields

$$\tau\Delta' = \frac{f_m}{f_l} - \Delta' = \frac{1}{\lambda}\left[\frac{1}{\varsigma}\left(1 - \frac{\varphi}{q}\right)\phi' + (1 - \sigma)\frac{\nu}{P}\right]. \tag{41}$$

This results can be rephrased as follows:

Proposition 3 *Carbon energy must be taxed at the rate*

$$\tau = \frac{1}{\Delta'\lambda}\left[(1 - \sigma)\frac{\nu}{P} + \frac{1}{\varsigma}\left(1 - \frac{\varphi}{q}\right)\phi'\right], \tag{42}$$

where λ is the marginal cost of labor, $\Delta'\lambda$ the marginal cost of carbon energy, σ the constant rate of relative risk aversion, ν the elasticity of consumption with respect to pollution, and $\frac{1}{\varsigma}\left(1 - \frac{\varphi}{q}\right)\phi'$ is the flow of the expected marginal relative disutility of pollution through the disaster, discounted by the expected effective rate of time preference, ς. The tax (42) consists of two parts:

$\frac{1}{\Delta' \lambda}(1-\sigma)\frac{\nu}{P}$ *the Pigouvian tax to correct the immediate damages of pollution.*

$\frac{1}{\Delta' \lambda}\frac{1}{\varsigma}\left(1-\frac{\varphi}{q}\right)\phi'$ *the precautionary tax to be applied before any disaster occurs.*

5 Concluding remarks

This document integrates technological change into an economy anticipating an environmental disaster.The extraction of carbon energy through labor contributes to pollution, increasing the risk of welfare-harming environmental disaster. Monopolists can produce intermediate goods with two alternative technologies: either brown technology relying on labor and carbon energy, or green technology relying solely on labor. R&D firms strive to displace incumbent monopolists by enhancing efficiency through labor.

In this setup, the green shift means a transfer of monopolists from the brown into the green technology. The central issue is how a government can improve welfare through taxation. To implement optimal policies, natural scientists should provide estimates of the pollution absorption rate, the relative environmental degradation at the time of the disaster, and the marginal effect of pollution on disaster risk.

When pollution incrementally affects welfare, policies can be implemented concurrently. However, when pollution impacts welfare through stochastic low-frequency shocks, the precautionary principle must be applied: policies must be enacted before the shock occurs, as it is too late once the shock happens. Consequently, the optimal tax on carbon energy must consist of two parts. The first part acts as a Pigouvian tax, addressing the immediate effects of pollution. It is proportional to the elasticity of consumption with respect to pollution. The second, precautionary part of the optimal tax counters the expected disaster due to pollution. It is proportional to the flow of the expected marginal relative disutility of pollution through the disaster, discounted by the expected effective rate of time preference. Since this two-part tax alone eliminates the distortion caused by pollution, R&D should neither be taxed nor subsidized.

Appendix

1. The government's problem

The government's maximizes utility (15) with (1) by $(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s})$ subject to the following constraints [cf., (2), (3), (11), (14), (32), (33) and (34)]:

$$c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P) \doteq P^{-\nu} y = P^{-\nu} \bigg(\int_{j \in I_b} y_{bj}^{1-1/\varepsilon} dj + \int_{j \in I_g} y_{gj}^{1-1/\varepsilon} dj \bigg)^{\varepsilon/(\varepsilon-1)}, \tag{43}$$

$$y_{bj} = a_b \gamma(j) f(l_j, m_j), \quad y_{gj} = a_g \mu(j) z_j, \tag{44}$$

$$\dot{a}_i \doteq \frac{da_i}{dt} = \delta \Omega_i(a_i, a_{-i}) \int_{j \in I_i} s_{ij} dj, \quad a_i(T) = a_i^T, \quad i \in \{b, g\}, \tag{45}$$

$$\dot{P} = \int_{j \in I_b} m_j dj - \beta P, \tag{46}$$

$$q = \begin{cases} \varphi \in (0,1) & \text{with probability } \phi(P) \in (0,1), \\ 1 & \text{with probalility } 1 - \phi(P), \end{cases} \quad \phi' > 0, \tag{47}$$

$$L = \int_{j \in I_g} z_j dj + \int_{j \in I_b} l_j dj + \Delta \bigg(\int_{j \in I_b} m_j dj \bigg) + \int_{j \in I_b} \frac{s_{bj}^2}{2\alpha} dj + \int_{j \in I_g} \frac{s_{gj}^2}{2\alpha} dj, \tag{48}$$

where $J = \int_{j \in I_b} dj$. The *value function* of this problem is

$$\Phi(a_b, a_g, P, q, T) \doteq \max_{\substack{(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}) \\ \text{s.t. (43)-(48)}}} \int_T^\infty u\big(c(\mathbf{z}, \mathbf{l}, \mathbf{m}, a_b, a_g), q\big) e^{\rho(T-t)} dt. \tag{49}$$

2. The Bellman equation

The Bellman equation for the maximization in (49) is given by

$$\rho \Phi(a_b, a_g, P, q, T) = \max_{\substack{\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s} \\ \text{s.t. (48)}}} \Psi(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}, a_b, a_g, P, q, T) \ \text{ with} \tag{50}$$

$$\begin{aligned} &\Psi(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}, a_b, a_g, P, q, T) \\ &\doteq u\big(c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P), q\big) + \phi(P) \big(\Phi\big|_{q=\varphi} - \Phi \big) + \frac{\partial \Phi}{\partial a_b} \dot{a}_b + \frac{\partial \Phi}{\partial a_g} \dot{a}_g + \frac{\partial \Phi}{\partial P} \dot{P} \\ &= \frac{q}{1-\sigma} c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P)^{1-\sigma} + \frac{\partial \Phi}{\partial a_b} \delta \Omega_b(a_b, a_g) \int_{j \in I_b} s_{bj} dj \\ &\quad + \frac{\partial \Phi}{\partial a_g} \delta \Omega_g(a_g, a_b) \int_{j \in I_g} s_{gj} dj + \frac{\partial \Phi}{\partial P} \bigg(\int_{j \in I_b} m_j dj - \beta P \bigg) + \phi(P) \big(\Phi\big|_{q=\varphi} - \Phi \big), \end{aligned} \tag{51}$$

where q jumps from 1 down to φ and $\phi(P)\big(\Phi\big|_{q=\varphi} - \Phi\big)$ vanishes at the moment of the environmental shock.

Because, by (51), there is a unique threshold production line

$$J = \arg \max_{J \text{ s.t. } (48)} \Psi(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}, a_b, a_g, P, q, T).$$

then $I_b = [0, J]$ and $I_g = [J, 1]$ hold true. Thus, the function (43) and the equations (48) and (51) can be rewritten as follows:

$$\begin{aligned} &\Psi(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}, a_b, a_g, P, q, T) = \\ &\frac{q}{1-\sigma} c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P) + \frac{\partial \Phi}{\partial a_b} \delta \Omega_b(a_b, a_g) \int_0^J s_{bJ} dj \\ &+ \phi(P)\big(\Phi\big|_{q=\varphi} - \Phi\big) + \frac{\partial \Phi}{\partial a_g} \delta \Omega_g(a_g, a_b) \int_J^1 s_{gJ} dj + \frac{\partial \Phi}{\partial P}(mJ - \beta P) \text{ with } \end{aligned} \quad (52)$$

$$\begin{aligned} &c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P) \doteq P^{-\nu} y = P^{-\nu} \left(\int_0^J y_{bj}^{1-1/\varepsilon} dj + \int_J^1 y_{gj}^{1-1/\varepsilon} dj \right)^{\varepsilon/(\varepsilon-1)} \\ &= P^{-\nu} \left[a_b^{1-1/\varepsilon} \int_0^J f(l_j, m_j)^{1-1/\varepsilon} dj + a_g^{1-1/\varepsilon} \int_J^1 z_j^{1-1/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)}, \end{aligned} \quad (53)$$

$$L = \int_J^1 z_j dj + \int_0^1 l_j dj + \Delta\left(\int_0^1 m_j dj \right) + \int_0^j \frac{s_{bj}^2}{2\alpha} dj + \int_J^1 \frac{s_{gj}^2}{2\alpha} dj. \quad (54)$$

The function (53) has partial derivatives

$$P^\nu \frac{\partial c}{\partial a_b} = \int_0^J \frac{\partial y}{\partial y_{bj}} \frac{\partial y_{bj}}{\partial a_b} dj = \int_0^J \left(\frac{y}{y_{bj}} \right)^{1/\varepsilon} \frac{y_{bj}}{a_b} dj = \frac{y^{1/\varepsilon}}{a_b} \int_0^J y_{bj}^{1-1/\varepsilon} dj, \quad (55)$$

$$P^\nu \frac{\partial c}{\partial a_g} = \int_J^1 \frac{\partial y}{\partial y_{gj}} \frac{\partial y_{gj}}{\partial a_g} dj = \int_J^1 \left(\frac{y}{y_{gj}} \right)^{1/\varepsilon} \frac{y_{gj}}{a_g} dj = \frac{y^{1/\varepsilon}}{a_g} \int_J^1 y_{gj}^{1-1/\varepsilon} dj, \quad (56)$$

$$P^\nu \frac{\partial c}{\partial J} = \frac{\partial y}{\partial J} = \frac{\varepsilon y^{1/\varepsilon}}{\varepsilon - 1} \big(y_{bJ}^{1-1/\varepsilon} - y_{gJ}^{1-1/\varepsilon} \big), \quad (57)$$

$$\frac{\partial c}{\partial a_b} a_b + \frac{\partial c}{\partial a_g} a_g = P^{-\nu} y^{1/\varepsilon} \underbrace{\left[\int_0^J y_{bj}^{1-1/\varepsilon} dj + \int_J^1 y_{gj}^{1-1/\varepsilon} dj \right]}_{= y^{1-1/\varepsilon}} = P^{-\nu} y = c. \quad (58)$$

The government maximizes the function (53) by $(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s})$ subject to (54). The Lagrangean for this is

$$\begin{aligned} &\mathcal{L} \doteq \Psi(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}, a_b, a_g, P, q, T) \ + \\ &\lambda \left[L - \int_J^1 z_j dj - \int_0^J l_j dj - \Delta\left(\int_0^J m_j dj \right) - \int_0^J \frac{s_{bj}^2}{2\alpha} dj - \int_J^1 \frac{s_{gj}^2}{2\alpha} dj \right], \end{aligned} \quad (59)$$

where the Lagrangean multiplier satisfies the condition

$$\lambda\left[L-\int_J^1 z_j dj-\int_0^J l_j dj-\Delta\left(\int_0^J m_j dj\right)-\int_0^J \frac{s_{bj}^2}{2\alpha}dj-\int_J^1 \frac{s_{gj}^2}{2\alpha}dj\right]=0. \tag{60}$$

The first-order conditions for maximizing (59) by $(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s})$ are

$$\lambda=\frac{\partial\Psi}{\partial l_j}=qc^{-\sigma}\frac{\partial c}{\partial l_j}=qc^{-\sigma}P^{-\nu}\frac{\partial y}{\partial y_b}a_b\gamma(j)f_l(l_j,m_j)\ \text{ for }\ j\in[0,J], \tag{61}$$

$$\lambda\Delta'=\frac{\partial\Psi}{\partial m_j}=qc^{-\sigma}\frac{\partial c}{\partial m_j}+\frac{\partial\Phi}{\partial P}=\underbrace{qc^{-\sigma}P^{-\nu}\frac{\partial y}{\partial y_b}a_b\gamma(j)}_{=\lambda/f_l(l_j,m_j),\ \ cf.,\ \ (61)}f_m(l_j,m_j)+\frac{\partial\Phi}{\partial P}$$

$$=\lambda\frac{f_m(l_j,m_j)}{f_l(l_j,m_j)}+\frac{\partial\Phi}{\partial P}\ \ \Leftrightarrow\ \ \frac{f_m(l_j,m_j)}{f_l(l_j,m_j)}=\Delta'-\frac{1}{\lambda}\frac{\partial\Phi}{\partial P}, \tag{62}$$

$$\lambda=\frac{\partial\Psi}{\partial z_j}=qc^{-\sigma}\frac{\partial c}{\partial z_j}=qc^{-\sigma}P^{-\nu}\frac{\partial y}{\partial y_g}\frac{\partial y_g}{\partial z_j}=qc^{-\sigma}P^{-\nu}\frac{\partial y}{\partial y_g}a_g\mu(j), \tag{63}$$

$$\frac{\lambda}{\alpha}s_{bj}=\frac{\partial\Psi}{\partial s_{bj}}=\frac{\partial\Phi}{\partial a_b}\delta\Omega_b(a_b,a_g),\ \ j\in[0,J], \tag{64}$$

$$\frac{\lambda}{\alpha}s_{gj}=\frac{\partial\Psi}{\partial s_g}=\frac{\partial\Phi}{\partial a_g}\delta\Omega_g(a_g,a_b),\ \ j\in[J,1], \tag{65}$$

$$\lambda\left(l_J+\Delta' m_J-z_J+\frac{s_{bJ}^2-s_{gJ}^2}{2\alpha}\right)=\frac{\partial\Psi}{\partial J}$$

$$=qc^{-\sigma}\frac{\partial c}{\partial J}+m_J\frac{\partial\Phi}{\partial P}+\underbrace{\frac{\partial\Phi}{\partial a_b}\delta\Omega_b}_{=\lambda s_{bJ}/\alpha\ \ cf.,\ \ (64)}s_{bJ}-\underbrace{\frac{\partial\Phi}{\partial a_g}\delta\Omega_g}_{=\lambda s_{gJ}/\alpha\ \ cf.,\ \ (65)}s_{gJ}$$

$$=qc^{-\sigma}\frac{\varepsilon y^{1/\varepsilon}}{\varepsilon-1}\left(y_{bJ}^{1-1/\varepsilon}-y_{gJ}^{1-1/\varepsilon}\right)+m_J\frac{\partial\Phi}{\partial P}+\frac{\lambda}{\alpha}(s_{bJ}^2-s_{gJ}^2)\ \ \Leftrightarrow$$

$$qc^{-\sigma}\frac{\varepsilon y^{1/\varepsilon}}{\varepsilon-1}\left(y_{bJ}^{1-1/\varepsilon}-y_{gJ}^{1-1/\varepsilon}\right)$$

$$=\lambda\left(l_J+\Delta' m_J-z_J+\frac{s_{bJ}^2-s_{gJ}^2}{2\alpha}+\frac{s_{gJ}^2-s_{bJ}^2}{\alpha}\right)-m_J\frac{\partial\Phi}{\partial P}$$

$$=\lambda\left(l_J-z_J+\frac{s_{gJ}^2-s_{bJ}^2}{2\alpha}\right)+\left(\underbrace{\Delta'-\frac{1}{\lambda}\frac{\partial\Phi}{\partial P}}_{=f_m/f_l\ \ cf.,\ \ (62)}\right)\lambda m_J$$

$$=\lambda\left(l_J+\frac{f_m}{f_l}m_J-z_J+\frac{s_{gJ}^2-s_{bJ}^2}{2\alpha}\right). \tag{66}$$

3. The stationary state

The relative efficiency of the green sector can be defined as follows:

$$a \doteq a_g/a_b. \tag{67}$$

The system (16), (44), (46), (53)-(65) and (67), has a stationary state where pollution P, the relative efficiency of the green sector, a, the inputs for production, $(\mathbf{z}, \mathbf{l}, \mathbf{m})$, the relative size of the brown sector, J, and inputs to R&D, $\mathbf{s}$, are constants, while consumption c, outputs (y, y_b, y_g) and efficiencies (a_b, a_g) grow at the constant rate G [cf., (8), (16), (45), (46) and (67)]:

$$(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, \mathbf{s}) \text{ constants}, \quad \dot{P} = 0 \;\Leftrightarrow\; \int_0^J m_j dj = \beta P, \tag{68}$$

$$\begin{aligned}
G &= \frac{\dot{c}}{c} = \frac{\dot{a}_b}{a_b} = \delta\Omega_b\left(1, \frac{a_g}{a_b}\right)\int_0^J s_{bj} dj = \delta\Omega_b(1,a)\int_0^J s_{bj} dj = \frac{\dot{a}_g}{a_g} \\
&= \delta\Omega_g\left(1, \frac{a_b}{a_g}\right)\int_J^1 s_{gj} dj = \delta\Omega_g\left(1, \frac{1}{a}\right)\int_J^1 s_{gj} dj \;\Leftrightarrow \\
&\Omega_b(1,a)\int_0^J s_{bj} dj = \Omega_g\left(1, \frac{1}{a}\right)\int_J^1 s_{gj} dj = \frac{G}{\delta}.
\end{aligned} \tag{69}$$

Differentiating the logarithm of equation (67) with respect to time t and noting (8) and (45) shows that the stationary state (69) is stable:

$$\begin{aligned}
\frac{\dot{a}}{a} &= \frac{\dot{a}_g}{a_g} - \frac{\dot{a}_b}{a_b} = \delta\Omega_g\left(1, \frac{1}{a}\right)\int_J^1 s_{gj} dj - \delta\Omega_b(1,a)\int_0^J s_{bj} dj \\
&= \delta\left[\Omega_g(1, 1/a)\int_J^1 s_{gj} dj - \Omega_b(1,a)\int_0^J s_{bj} dj\right] \\
&\text{with } \frac{\partial}{\partial a}\left(\frac{\dot{a}}{a}\right)_{\dot{a}=0} = -\delta\left(\underbrace{\frac{\partial \Omega_g}{\partial a_b}}_{+}\underbrace{\frac{1}{a^2}\int_J^1 s_{gj} dj}_{\text{constant}} + \underbrace{\frac{\partial \Omega_b}{\partial a_g}}_{+}\underbrace{\int_0^J s_{bj} dj}_{\text{constant}}\right) < 0.
\end{aligned}$$

The constraint $\beta P = \int_0^J m_j dj$ defines the function $J(P, \mathbf{m})$. Differentiating this function totally yields $\beta\, dP = m_J dJ + \int_0^J dm_j\, dj$, by which the function can be presented with the partial derivatives as follows:

$$J(P, \mathbf{m}), \quad \frac{\partial J}{\partial P} = \frac{\beta}{m_J}, \quad \frac{\partial J}{\partial m_j} = -\frac{1}{m_J} \text{ for } j \in [0, J]. \tag{70}$$

4. The specification of the value function

The government's problem is addressed by identifying a specification for the value function (49) that satisfies the Bellman equation (50) along with (52) in the stationary state (68) and (69). Let's propose that (49) represents the maximal periodic utility $u = \frac{q}{1-\sigma} c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g)^{1-\sigma}$ subject to the equilibrium condition of the labor market, (54), and the stationary-state conditions (68) and (69), divided by a positive and piecewise differentiable function ς of state variables (P, q) [cf., (1), (43), (49) and (70)]:

$$\begin{aligned}\Phi\big(a_b, a_g, P, q, T\big) &\doteq \max_{(\mathbf{z}, \mathbf{l}, \mathbf{m}, J) \text{ s.t. } (54),(68),(69)} \frac{u}{\varsigma(P, q)} \\ &= \max_{(\mathbf{z}, \mathbf{l}, \mathbf{m}, J) \text{ s.t. } (54),\ \beta P = \int_0^J m_j dj} \frac{q/(1-\sigma)}{\varsigma(P, q)} c(\mathbf{z}, \mathbf{l}, \mathbf{m}, J, a_b, a_g, P)^{1-\sigma} \\ &= \frac{q/(1-\sigma)}{\varsigma(P, q)} P^{(\sigma-1)\nu} \Big[\max_{(\mathbf{z}, \mathbf{l}, \mathbf{m}) \text{ s.t. } (54)} y\big(\mathbf{z}, \mathbf{l}, \mathbf{m}, J(P, \mathbf{m}), a_b, a_g\big)\Big]^{1-\sigma}.\end{aligned} \tag{71}$$

By (70), the partial derivatives of the function (71) with respect to the state variables are the following:

$$\frac{\partial \Phi}{\partial a_b} = (1-\sigma)\frac{\Phi}{c}\frac{\partial c}{\partial a_b}, \quad \frac{\partial \Phi}{\partial a_g} = \frac{\Phi}{c}\frac{\partial c}{\partial a_g} = (1-\sigma)\frac{\Phi}{c}\frac{\partial c}{\partial a_g}, \tag{72}$$

$$\begin{aligned}\frac{\partial \Phi}{\partial P} &= (1-\sigma)\frac{\Phi}{y}\frac{\partial y}{\partial J}\frac{\partial J}{\partial P} - \frac{\Phi}{\varsigma}\frac{\partial \varsigma}{\partial P} + (\sigma-1)\nu\frac{\Phi}{P} \\ &= (1-\sigma)\frac{\Phi}{y}\frac{\partial y}{\partial J}\frac{\beta}{m_J} - \frac{\Phi}{\varsigma}\frac{\partial \varsigma}{\partial P} - (1-\sigma)\nu\frac{\Phi}{P}.\end{aligned} \tag{73}$$

By (58) and (72), the function (71) has the property

$$\frac{\partial \Phi}{\partial a_b}a_b + \frac{\partial \Phi}{\partial a_g}a_g = \frac{q}{\varsigma}c^{-\sigma}\underbrace{\left(\frac{\partial c}{\partial a_b}a_b + \frac{\partial c}{\partial a_g}a_g\right)}_{=c} = \frac{q}{\varsigma}c^{1-\sigma} = (1-\sigma)\Phi. \tag{74}$$

At the occurrence of the shock, q falls down from 1 to φ, but the state variables (a_b, a_g, P) do not change. Hence, by (31) and (71), one obtains

$$\frac{\Phi - \Phi\big|_{q=\varphi}}{\Phi} = \frac{u - u\big|_{q=\varphi}}{u} = 1 - \frac{\varphi}{q} > 0. \tag{75}$$

Inserting the stationary-state conditions (68) and (69), the result (74) and the jump condition (75) into the Bellman equation (50) with (52) and dividing by Φ solve for ς:

$$\rho = \frac{\Psi}{\Phi} = \frac{u}{\Phi} + \phi(P)\frac{\Phi\big|_{q=\varphi} - \Phi}{\Phi} + \frac{1}{\Phi}\frac{\partial \Phi}{\partial a_b}\dot{a}_b + \frac{1}{\Phi}\frac{\partial \Phi}{\partial a_g}\dot{a}_b + \frac{\partial \Phi}{\partial P}\underbrace{\dot{P}}_{=0}$$

$$= \frac{u}{\Phi} + \phi(P)\underbrace{\frac{\Phi\big|_{q=\varphi} - \Phi}{\Phi}}_{=\varphi/q-1} + \underbrace{\left(\frac{a_b}{\Phi}\frac{\partial \Phi}{\partial a_b} + \frac{a_g}{\Phi}\frac{\partial \Phi}{\partial a_g}\right)}_{=1-\sigma}G$$

$$= \varsigma + \phi(P)(\varphi/q - 1) + (1-\sigma)G \;\; \Leftrightarrow \;\; \varsigma = \rho + (\sigma - 1)G + \phi(P)(\varphi/q - 1). \tag{76}$$

By this, the function $\varsigma(P, q)$ can be specified as

$$\varsigma(P, q) \doteq \rho + (\sigma - 1)G + \phi(P)\Big(1 - \frac{\varphi}{q}\Big) \text{ with } \frac{\partial \varsigma}{\partial P} = \Big(1 - \frac{\varphi}{q}\Big)\phi' > 0. \tag{77}$$

Because $q \in \{\varphi, 1\}$, the function (77) is piecewise differentiable, satisfying the Bellman equation (50) with (52) in the stationary state (68) and (69). From (71), (73) and (77) it follows that

$$\frac{1}{\Phi}\frac{\partial \Phi}{\partial P} = \frac{1-\sigma}{y}\frac{\partial y}{\partial J}\frac{\beta}{m_J} - \frac{1}{\varsigma}\Big(1 - \frac{\varphi}{q}\Big)\phi' - (1-\sigma)\frac{\nu}{P}. \tag{78}$$

6. Optimal policy

By (56) and (72), the first-order conditions (65) can be written as follows:

$$\frac{\int_0^J s_{bj}dj}{\int_J^1 s_{gk}dk} = \frac{\frac{\partial \Phi}{\partial a_b}a_b}{\frac{\partial \Phi}{\partial a_g}a_g} = \frac{\frac{\partial c}{\partial a_b}a_b}{\frac{\partial c}{\partial a_g}a_g} = \frac{\int_0^J y_{bj}^{1-1/\varepsilon}dj}{\int_J^1 y_{gk}^{1-1/\varepsilon}dk}. \tag{79}$$

Plugging (78) into the first-order condition (62) yields

$$\frac{f_m(l_j, m_j)}{f_l(l_j, m_j)} - \Delta' = -\frac{1}{\lambda}\frac{\partial \Phi}{\partial P} = \frac{1}{\lambda}\left[\frac{\sigma - 1}{y}\frac{\partial y}{\partial J}\frac{\beta}{m_J} + \frac{1}{\varsigma}\Big(1 - \frac{\varphi}{q}\Big)\phi' + (1-\sigma)\frac{\nu}{P}\right] \tag{80}$$

for $j \in [0, J]$. Results (61) and (63) are summarized in (35). The result (66) corresponds to (36), the result (76) corresponds to (37), the result (79) corresponds to (38), and the result (80) corresponds to (39).

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