Helsinki GSE

HELSINKI GSE DISCUSSION PAPERS 2 · 2023

On Robustness of Average Inflation Targeting

Seppo Honkapohja
Nigel McClung

Helsinki GSE Discussion Papers

Helsinki GSE Discussion Papers 2 · 2023

Seppo Honkapohja and Nigel McClung:
On Robustness of Average Inflation Targeting

ISBN 978-952-7543-01-6 (PDF)
ISSN 2954-1492

Helsinki GSE Discussion Papers:
https://www.helsinkigse.fi/discussion-papers

Helsinki Graduate School of Economics
PO BOX 21210
FI-00076 AALTO
FINLAND

Helsinki, February 2023

ON ROBUSTNESS OF AVERAGE INFLATION TARGETING*

SEPPO HONKAPOHJA NIGEL MCCLUNG†

February 5, 2023

Abstract

This paper considers average inflation targeting (AIT) policy in a New Keynesian model with adaptive learning agents. Our analysis raises concerns regarding robustness of AIT when agents have imperfect knowledge. In particular, the target steady state may not be robustly stable under learning if the length of the averaging window is not public knowledge. Near the low steady state with interest rates at the zero lower bound, AIT does not necessarily outperform standard inflation targeting policy. Policymakers can improve outcomes under AIT by communicating the averaging window, or using an asymmetric rule that responds more aggressively to below-target average inflation.

Keywords: Adaptive Learning, Inflation Targeting, Zero Interest Rate Lower Bound.

JEL codes: E31, E52, E58

1 Introduction

The period from 2008 to 2020 was a very challenging time for macroeconomic policy-making and in particular for monetary policy.[1] In most years since 2008 central banks have had to keep the policy interest rates at approximately zero level, popularly called the zero lower bound (ZLB) or the liquidity trap. The usual framework of inflation targeting, which was the initial monetary policy strategy in the global financial crisis, became largely ineffective in the ZLB regime. Once policy rates were effectively down to the ZLB level, central

*We are grateful to Klaus Adam, Markus Haavio, Sebastian Schmidt, and our discussants, Lilia Maliar, Bruce Preston, and Jacek Suda, conference and seminar participants at the 2021 NBU-NBP Annual Research Conference, 2021 Bank of Israel-CEPR conference, 2021 Bank of Finland-CEPR conference, the 2021 Expectations in Dynamic Macroeconomic Models conference, 24th Central Bank Macroeconomic Modeling Workshop, and at seminars at ISER Osaka University, Helsinki GSE and Bank of Finland for helpful comments. First author acknowledges financial support from the Yrjö Jahnsson Foundation. The views expressed in this paper are not necessarily the views of the Bank of Finland.

†The author for correspondence. Address: Snellmaninaukio, Helsinki, FI 00101. Email: nigel.mcclung@bof.fi.

[1]We do not consider the challenges from the Covid 19 pandemic.

banks had to revert to unconventional monetary policies which took the form of liquidity operations, credit easing, large scale asset purchases and forward guidance about future course of policy. A number of empirical studies have shown that these new policies had qualitatively the right kind of macroeconomic effects, but the estimated magnitudes have been variable.[2]

The ZLB regime after the financial crisis inspired discussions among prominent central bankers and academics about possible reform of the monetary policy framework. Alternatives to inflation targeting were explored. Price level targeting (PLT) and the related concept of nominal GDP targeting were perhaps the most widely discussed suggestions for a more appropriate monetary policy framework. Evans (2012) and Carney (2012) were among the first commentators and more recently, for example, Williams (2017), Bernanke (2017) and Bullard (2018) suggested that PLT and more complex "switching policies" should be studied further. In 2019 the Federal Reserve initiated a review of its monetary policy strategy. The review process culminated in August 2020 in the announcement by the Chairman Powell (2020) that the policy framework of the Fed is to be based on Average Inflation Targeting (AIT).[3]

AIT has not been widely studied in the research literature. Nessen and Vestin (2005) studied a rational expectations (RE) model in which the central bank targets a simple moving average of inflation rates. Reifschneider and Williams (2000) suggested a bookkeeping device to keep track of deviations between the actual interest rate and a reference rate based on the Taylor rule. Mertens and Williams (2019) use the interest rate from the optimal policy under discretion and RE (when the ZLB can be binding) as a reference rate which is combined with AIT. Budianto, Nakata and Schmidt (2023) study the optimal policy under discretion when agents are myopic and the central bank loss function incorporates an exponential moving average of actual inflation rates. Amano, Gnocchi, Leduc and Wagner (2020) study the optimal averaging window under AIT in a model with rule-of-thumb price-setters. Jia and Wu (2022) consider the role of ambiguities in communication with AIT. Andrade, Gali, Le Bihan and Matheron (2021) is a recent applied paper that focuses on alternative make-up strategies for monetary policy.[4]

Most of the above mentioned papers assume that agents understand the basic structure of AIT policy. This assumption is hard to reconcile with recent studies on how expectations respond to AIT. Coibion, Gorodnichenko, Knotek II and Schoenle (2020) found that the Fed's August 2020 announcement of AIT had little effect on household inflation expectations. This result could suggest that households do not understand the basic structure and implications of AIT. Salle (2021) examined data from laboratory experiments involving AIT and finds that agents struggle to understand the lag structure implied by AIT. On the other hand, Hoffman, Moench, Pavlova and Schultefrankenfeld (2022) find that German households might understand the implications of an asymmetric AIT strategy.

In this paper we consider the performance of AIT in a standard New Keynesian (NK) model when private agents have imperfect knowledge of the economy and have to engage in learning to forecast its dynamics. It is assumed that when forming expectations private agents statistically estimate the laws of motion for the endogenous variables that they need to forecast. In this setting private agents make in each period optimal decisions given the

[2]See Moessner, Jansen and de Haan (2017), Dell'Ariccia, Rabanal and Sandri (2018), Kuttner (2018) and Bhattarai and Neely (2016) for reviews of the empirical literature on unconventional monetary policy.

[3]See e.g. Svensson (2020) for a wide-ranging discussion of alternatives that were considered by the Fed.

[4]There are also related studies by the staff of the Federal Reserve System that were done as part of the Fed strategy review, such as Hebden, Herbst, Tang, Topa and Winkler (2020).

current forecasts and the economy evolves as a sequence of temporary equilibria that are defined by forecasts and private decisions in each period. As time progresses, new data leads to updating of the forecast functions and new temporary equilibria. This approach is called adaptive learning, and it relies on a more realistic model of expectations formation and decision making than rational or boundedly rational decision rules under RE.[5]

The induced learning behavior influences the actual dynamics of endogenous variables. In benign circumstances the economy reaches a long-run RE equilibrium, but this depends on the structure of the economy and in particular the policy rule used by the central bank. As a starting point, transparency about the averaging window is considered. If the central bank makes a credible commitment to a transparent averaging window, then convergence to the target steady state occurs if the model parameters correspond to those used in studies of the usual Taylor and related rules.[6] However, when AIT is practiced under opacity of its details, which is arguably the current framework of the Fed,[7] convergence to long run equilibrium may not take place, so that the form of monetary policy can play a crucial role in guaranteeing long-run convergence.

Our analysis raises warning signals as regards robustness of economic performance with AIT policy in conditions of imperfect knowledge and learning. In the main case it is assumed that AIT is practiced under opacity of its details. Specifically, we assume agents do not know the length of the averaging window and other details of a symmetric Taylor-type policy rule. In this setup the outcome is precarious in that local convergence of learning to target steady state depends on stickiness of prices. With full price flexibility there is local instability while with price stickiness the steady state is locally stable but not robustly so: stability holds only if the speed of learning, i.e. the rate at which agents update their expectations as new data becomes available, is implausibly slow. Moreover, using simulations from a non-linear New Keynesian model with a zero lower bound constraint indicate that even a transparent AIT regime does not necessarily outperform a standard Taylor rule at the lower bound on interest rates.

Given that transparency may have modest benefits, and requires the central bank to commit to an averaging window, we next investigate an asymmetric strategy in which the central bank aims to overshoot the inflation target, but does not aim to undershoot the target. We find that an asymmetric rule can improve outcomes under conditions of opacity. Other variations, including weighted average schemes, are also considered, and directions for future work are discussed.

In the next section we introduce the AIT formulation and illustrate the basic results using a simple model. The main model is developed in Section 2. Sections 3 through 6 present the main stability results and consider various alterations to the AIT framework. Proofs of the results and various modelling details are in the several appendices.

1.1 Introductory Example

The basic idea behind average inflation targeting (AIT) is that, when comparing actual inflation against its long-run target, the measure of actual inflation is an average of past

[5]For example, IMF World Economic Outlook, October 2022 uses adaptive learning as the baseline framework for modelling expectations formation. See IMF (2022), pp.63-66 and the Annex to the report.

[6]Evans and Honkapohja (2003) and Evans and Honkapohja (2006) showed how the form of interest rate rule for implementing optimal policy can be crucial to ensure convergence to REE. Bullard and Mitra (2002) studied local convergence conditions when a Taylor rule is applied.

[7]See e.g. the interview of John Williams in FT Live, November 13, 2020.

inflation rates. This average inflation rate is the key indicator for policy decisions, so that policy is tightened vs. loosened if the average of past inflation rates is above vs. below respectively its target value. There are of course different ways to measure average value and there is also the issue of length of the data window in its computation. In addition, major issues about communication of the AIT framework to the private economy are relevant.

A simple example is now used to illustrate that introducing AIT under conditions of imperfect knowledge can create stability concerns in the resulting economic dynamics. Consider the following linearized New Keynesian model[8]

$$\hat{y}_t = \hat{y}_t^e - \tilde{\sigma}(\beta \hat{R}_t - \hat{\pi}_t^e), \tag{1}$$

$$\hat{\pi}_t = \beta \hat{\pi}_t^e + \kappa \hat{y}_t, \tag{2}$$

where $\hat{y}$ is the output gap, $\hat{\pi}$ is inflation, $\hat{R}$ is the nominal interest rate, $\hat{\pi}_t^e = E_t^* \hat{\pi}_{t+1}$, $\hat{y}_t^e = E_t^* \hat{y}_{t+1}$, and $\tilde{\sigma}$ is proportional to the intertemporal elasticity of substitution. Notation $\hat{x} = x - x^*$ gives the deviation from the target steady state. Note that κ is decreasing in price rigidity and prices become totally flexible as $\kappa \to \infty$.

We assume the central bank sets the nominal interest rate in response to an average of deviations from the inflation target π^* in the past $L-1$ periods.[9]

$$\hat{R}_t = \psi \sum_{k=0}^{L-1} \frac{\hat{\pi}_{t-k}}{\pi^*}. \tag{3}$$

Note that the AIT rule (3) is a standard inflation targeting (IT) Taylor rule if $L = 1$, and it can be re-written as a Wicksellian price level targeting (PLT) rule as $L \to \infty$. Thus, the simple AIT rule nests traditional IT and PLT alternatives as limiting cases. For our purposes, we have an average inflation targeting central bank if $L > 1$ and L is finite.

We assume there is opacity about the AIT policy rule, (3), which means that agents do not understand the length of the averaging window, L, or other details of the policy rule. This is a natural starting point for the analysis, as the world's only average inflation targeting central bank, the Federal Reserve, has not communicated an averaging window since implementing AIT in 2020. If agents do not understand the lag structure of monetary policy decisions, they may omit the critical $L-1$ lags of inflation from their forecasting models (usually called the perceived law of motion or PLM). Specifically, assume that the agents might forecast inflation using a simple weighted average of past inflation, called steady state learning with constant gain, which formally is

$$\hat{\pi}_t^e = \hat{\pi}_{t-1}^e + \omega(\hat{\pi}_{t-1} - \hat{\pi}_{t-1}^e), \tag{4}$$

where $\omega > 0$ is a small constant.[10] Importantly, the steady state learning algorithm encodes agents' beliefs that lags of inflation do not matter for the current monetary policy stance. This way of forming inflation expectations would be natural under a standard inflation targeting regime (i.e. $L = 1$).

[8] The results in this paper do not depend on whether we linearize or log-linearize the model around the target steady state equilibrium.

[9] The inflation term could be divided by L, but this would not change the result as L can be incorporated into ψ.

[10] In the main paper we also consider cases where agents use fewer than L lags in the forecasting.

For the sake of simplicity, we assume output expectations are based on (2) and given as[11]

$$\hat{y}_t^e = \frac{1-\beta}{\kappa}\hat{\pi}_t^e. \tag{5}$$

If we substitute (5), (2), and (3) into (1) we have

$$\hat{\pi}_t = \frac{\pi^*\left(\kappa^{-1}+\tilde{\sigma}\right)}{\pi^*\kappa^{-1}+\beta\tilde{\sigma}\psi}\hat{\pi}_t^e - \frac{\beta\tilde{\sigma}\psi}{\pi^*\kappa^{-1}+\beta\tilde{\sigma}\psi}\sum_{k=1}^{L-1}\hat{\pi}_{t-k}. \tag{6}$$

By lagging (6) one period and solving for $\hat{\pi}_{t-1}^e$, then substituting this expression for $\hat{\pi}_{t-1}^e$ and resulting (4) into (6) we obtain for any κ

$$\begin{aligned}\hat{\pi}_t &= \frac{\pi^*\kappa^{-1}-\tilde{\sigma}\omega(\beta\psi-\pi^*)}{\beta\tilde{\sigma}\psi+\pi^*\kappa^{-1}}\hat{\pi}_{t-1} \\ &- \frac{\omega\beta\tilde{\sigma}\psi}{\pi^*\kappa^{-1}+\beta\tilde{\sigma}\psi}\sum_{k=2}^{L-1}\hat{\pi}_{t-k} + \frac{(1-\omega)\beta\tilde{\sigma}\psi}{\pi^*\kappa^{-1}+\beta\tilde{\sigma}\psi}\hat{\pi}_{t-L}.\end{aligned} \tag{7}$$

A key question of interest is whether the target steady state (i.e. $\hat{\pi}=0$) is locally stable under learning with opacity. In other words, will the process (7) converge for different values of L and sufficiently small values of the constant gain parameter, ω? We first consider the case of flexible prices.

Remark 1 *Assume the Taylor Principle is satisfied (i.e. $\psi > \beta^{-1}\pi^*$). In the limit $\kappa \to \infty$, the steady state π^* is locally stable under the system (7) if $L \leq 3$ but is explosive if $L = 4$ and for all higher, finite values of L.*

This remark can be proved by computing the characteristic polynomial to see when its roots are inside the unit circle, see Appendix D.2. We will give detailed proof using a more general model with flexible prices.

Figure I illustrates the instability with $L = 5$ and standard numerical values for the parameters: $\beta = 0.99$, $\psi = 1.5$ and $\omega = 0.001$. The initial conditions are $\hat{\pi}_0^e = 0.02$, $\hat{\pi}_0 = 0.09$, $\hat{\pi}_{-1} = 0.1$, $\hat{\pi}_{-2} = 0.001$, $\hat{\pi}_{-3} = 0.03$ and $\hat{\pi}_{-4} = 0.05$. Divergence is very slow as the gain parameter ω is very small.[12]

Figure I HERE

To get an intuition for Remark 1 and the diverging oscillations depicted in Figure I, notice that (7) becomes the following process as $\kappa \to \infty$

$$\hat{\pi}_t = \omega(\frac{\pi^*}{\beta\psi}-1)\hat{\pi}_{t-1} - \omega\sum_{i=2}^{L-1}\hat{\pi}_{t-i} + (1-\omega)\hat{\pi}_{t-L}. \tag{8}$$

Under a simple IT rule ($L = 1$) equilibrium inflation is a monotonic sequence, $\pi_t = (1 + \omega(\pi^*(\beta\psi)^{-1} - 1))\pi_{t-1}$, that converges slowly to steady state if $\beta^{-1}\pi^* < \psi$ and $\omega > 0$ is small. With AIT ($L > 1$), the sequence of inflation is no longer monotone. For example,

[11] Alternatively, we could assume that $\hat{y}_t^e$ is based on a steady state learning scheme akin to (4), but this would not affect the qualitative results in this section.

[12] Results from a long simulation for 50,000 periods showing divergence in the example of Figure I are available on request.

the AIT rule aims for "makeup" inflation ($\hat{\pi}_t > 0$) after an initial undershooting of the target ($\hat{\pi}_0 < 0$). Further, because the policymaker targets a finite moving average of inflation, the measure of *average inflation* can overshoot the target during a period of makeup inflation and this compels the policymaker to aim for a subsequent undershooting of the target steady state. Thus, the use of a finite moving average of inflation gives rise to the oscillatory pattern of over- and undershooting depicted in Figure I. With opacity, agents cannot forecast these temporary oscillations in inflation, and so their long-run inflation expectations drift and inflation becomes dynamically unstable.

From Remark 1, instability problems arise with *finite* values of L. We noted earlier that the AIT rule converges to a Wicksellian PLT rule in the limit $L \to \infty$.[13] In this case, the flexible price model becomes an AR(2) process for the price level: $\hat{p}_t = (1 + \omega(\pi^*(\beta\psi)^{-1} - 1))\,\hat{p}_{t-1} - \omega\pi^*(\beta\psi)^{-1}\hat{p}_{t-2} + \omega\hat{p}_0$. For small ω, $\hat{p}_t$ clearly converges as t increases, and therefore $\hat{\pi}_t \to 0$ for given $\hat{p}_0$. Thus, unstable dynamics emerge under AIT with opacity for sufficiently high, finite values of L, but not under IT or PLT. In other words, Remark 1 poses a unique problem for an average inflation targeting central bank. Under some conditions of imperfect knowledge, AIT is not an intermediate case between IT and PLT.

Similar results obtain for other under-parameterized PLMs. For instance, if agents forecast using a simple AR(1) model of inflation (i.e. $\pi_t = a + b\pi_{t-1}$) which they estimate recursively using a constant gain learning scheme, then their estimates of the intercept (a) will not converge for $L \geq 4$. To help distinguish between different under-parameterized PLMs, we use the terminology "full opacity" to describe learning with forecasting models that exclude *all* lags of inflation, such as the case of steady state learning with (4), and "opacity" describes learning with a forecasting model that includes at least one but less than $L - 1$ lags of inflation, π_{t-1}.

The instability result in Remark 1, however, does not hold up when prices are sticky (i.e. κ is small). When prices are very sticky, and ω is very small, then inflation is not very responsive to lags of inflation and consequently, (7) boils down to $\hat{\pi}_t \approx A\hat{\pi}_{t-1}$ with A slightly smaller than 1.

Remark 2 *Assume that $\psi > \beta^{-1}\pi^*$. If κ is small and ω is sufficiently close to zero, then the steady state π^* is stable under the system (7) for all L.*

Remarks 1 and 2 give contrasting results; AIT yields stability when there is price stickiness (provided that speed of learning, ω, is low), but is problematic if the economy has flexible prices. However, Remark 2 assumes ω is very small, and this begs important questions. In particular, with sticky prices, is the target steady state robustly stable for empirically plausible calibrations of ω, or do we need ω to be implausibly small? If the target steady state is not robustly stable, then what can a central bank do in order to successfully implement AIT? If the target steady is robustly stable, then can AIT succeed in bringing the economy out of a liquidity trap with binding ZLB? The remainder of the paper is devoted to these questions and related issues.

[13] In the limit $L \to \infty$, we have $R_t = \frac{\psi}{\pi^*}(\hat{p}_t - \hat{p}_0)$ where $\hat{p}_t = \hat{\pi}_t + \hat{p}_{t-1}$ is the current price level and $\hat{p}_0$ is the given initial price level.

2 New Keynesian Model

This section develops a standard New Keynesian model of learning. In the model, a continuum of household-firms produce a differentiated consumption good under monopolistic competition and price adjustment costs in the spirit of Rotemberg (1982). Agents optimize over the infinite horizon in accordance with the "anticipated utility" approach formulated by Kreps (1998) and discussed in Sargent (1999) and Cogley and Sargent (2008).

The utility and production functions are assumed to be identical and agents have homogenous point expectations, so that there is a representative agent.[14] Government uses monetary policy, buys a fixed amount of output and finances spending by taxes and issues of public debt. Monetary policy is conducted in terms of an interest rate rule in the cashless limit. It should be recalled that the nonlinear model we use has two steady states, the inflation-target and liquidity trap (or ZLB) steady states, when interest rate setting follows a suitable nonlinear Taylor rule entailing an active policy response at the target level of inflation. See e.g. Benhabib, Schmitt-Grohe and Uribe (2001) and Benhabib, Evans and Honkapohja (2014) for the RE and learning versions of the model.[15] The existence of two steady states is a key feature also with the PLT and AIT interest rate rules.

We note that a classical monetary model with full price flexibility is obtained from our model in the limit by setting the adjustment costs to zero. In this case the Phillips curve is replaced by a static first order condition for consumption and labor supply. The cases of price stickiness and price flexibility are both discussed below as the distinction turns out to be important for the results. For brevity, formal details of the two versions of the model are given in Appendix A.[16]

2.1 Behavioral Rules

The analysis relies on two behavioral rules of private agents: the Phillips curve and the consumption function. Starting with the former, the Phillips curve takes the form

$$\begin{aligned} Q_t \;=\; & \tilde{K}(y_t, y^e_{t+1}, y^e_{t+2}\ldots) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\frac{y_t}{(y_t-(\bar{g}+\tilde{g}_t))^{\sigma}} + \\ & \frac{\nu}{\gamma}\sum_{j=1}^{\infty}\alpha^{-1}\beta^j\left(y^e_{t+j}\right)^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\sum_{j=1}^{\infty}\beta^j\frac{y^e_{t+j}}{(y^e_{t+j}-(\bar{g}+\rho^j\tilde{g}_t))^{\sigma}}, \end{aligned} \tag{9}$$

where

$$Q_t = (\pi_t - 1)\,\pi_t. \tag{10}$$

Here y_t denotes output and $\bar{g}$, $\tilde{g}_t$ are the mean and random parts of government spending. ν, α, γ and β are parameters for substitution elasticity, labor input exponent in production, price adjustment costs and subjective discount rate, respectively. Prices are sticky if $\gamma > 0$ and price flexibility obtains in the limit $\gamma \to 0$. Superscript e indicates expectations while subscripts indicate the periods $t+j$, $j = 0, 1, 2, ...$

[14]Point expectations are an assumption of bounded rationality. It means that agents treat the conditional expectation of a nonlinear function of random variables as equal to the nonlinear function of the conditional expectations.

[15]With unchanged policy, there is a third interior steady state if public consumption enhances agents' utility, see Evans, Honkapohja and Mitra (2020).

[16]The derivation of model is also given in Benhabib et al. (2014) and Honkapohja and Mitra (2020).

The form (9) of the Phillips curve is special as expected future inflation does not directly affect current inflation. (There is an indirect effect via current output in the Phillip's curve.) This formulation is based on the assumption of a representative agent and a simplifying assumption about expectations, see Appendix A.2 for formal details. One could allow for heterogenous expectations along the learning path. The formulation facilitates the stability analysis without loss of generality in the results.

To derive the consumption function it is assumed for simplicity that consumers are Ricardian in the sense that they amalgamate their own intertemporal budget constraint and that of the government (where the latter is evaluated at price expectations of the consumer). It can be shown that the consumption function takes the form

$$c_t \sum_{j=1}^{\infty} \beta^{j\sigma^{-1}} (D^e_{t,t+j})^{(1-\sigma)\sigma^{-1}} = \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t)), \tag{11}$$

where c_t, $\pi_t = \frac{P_t}{P_{t-1}}$ and R_t denote private consumption, (gross) inflation rate and (gross) interest rate for loan from period t to $t+1$, respectively. The discount factor is

$$D^e_{t,t+j} = \frac{R_t}{\pi^e_{t+1}} \prod_{i=2}^{j} \frac{R^e_{t+i-1}}{\pi^e_{t+i}}. \tag{12}$$

In practice central banks do not make their policy instrument rules known. This is reflected in (12) as private agents must form expectations about future interest rates.

The aggregate demand function is obtained by combining the market clearing condition $y_t = c_t + g_t$ with the consumption function (11)

$$y_t = Y((y^e_{t+j}, \pi^e_{t+j}, R^e_{t+j}, \bar{g} + \rho^j \tilde{g}_t)_{j=1}^{\infty}, R_t, g_t). \tag{13}$$

2.2 Average Inflation Targeting (AIT)

The central bank uses an interest rate rule that depends on average inflation

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p[\frac{P_t - \bar{P}_{t,L}}{\bar{P}_{t,L}}] + \psi_y[\frac{y_t - y^*}{y^*}], 0], \tag{14}$$

$$\bar{P}_{t,L} = (\pi^*)^L P_{t-L} \text{ and} \tag{15}$$

$$\pi_t = \frac{P_t}{P_{t-1}}. \tag{16}$$

Here $\bar{P}_{t,L}$ denotes the target price level. It is formulated with a target level for inflation π^* and $\bar{P}_{t,L}$ is computed by compounding the actual price level L periods ago using target inflation rate π^*. Notice that (14) becomes a simple inflation targeting rule when $L = 1$. As $L \to \infty$, (14) becomes a Wicksellian PLT rule with inflation target path given by $\bar{P}_{t,\infty} = (\pi^*)^t P_0$ for all t.

The rule (15) implies that

$$\frac{P_t}{\bar{P}_{t,L}} = \frac{P_t}{P_{t-1}} \cdots \frac{P_{t-(L-1)}}{(\pi^*)^L P_{t-L}} = (\pi^*)^{-L} \prod_{i=0}^{L-1} \pi_{t-i},$$

so the basic AIT rule with the ZLB constraint can be written as

$$\begin{aligned} R_t &= R(y_t, \pi_t, ..., \pi_{t+1-L}) \\ &\equiv 1 + \max\left[\bar{R} - 1 + \psi_p\left[\prod_{i=0}^{L-1}\frac{\pi_{t-i}}{(\pi^*)^L} - 1\right] + \psi_y\left(\frac{y_t}{y^*} - 1\right), 0\right]. \end{aligned} \tag{17}$$

Rule (17) is the starting point of our analysis of average inflation targeting, but other variants will also be considered. Also note that (17) is a nonlinear version of (3).

2.3 Stability of the Target Steady State Under Learning

We start by considering as a benchmark the case where there is transparency about the averaging window. The nonlinear model comprised of the Phillips curve (9), the aggregate demand function (13) and the AIT interest rate rule (17). The model is first linearized around the target steady state by computing the general form of agents' linearized infinite-horizon (IH) decision rules. If agents are identical, the linearized IH Phillips curve obtained in Appendix A.4 is

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j}, \tag{18}$$

where $\hat{x}$ denotes a linearized variable, and κ is the slope of the Phillips curve. In Appendix A.4 it is shown that the linearized aggregate demand function takes the form

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty}\beta^j\left(\frac{1-\beta}{\beta}\hat{y}^e_{t+j} - \frac{c^*}{\sigma}\left(\beta\hat{R}^e_{t+j}(\pi^*)^{-1} - \hat{\pi}^e_{t+j}(\beta\pi^*)^{-1}\right)\right). \tag{19}$$

The linearized expression for the interest rate rule (17) near the target steady state is

$$\hat{R}_t = \psi_p \sum_{k=0}^{L-1}\frac{\hat{\pi}_{t-k}}{\pi^*} + \psi_y\frac{\hat{y}_t}{y^*}. \tag{20}$$

We note that the AIT interest rate rule induces lags of inflation into the otherwise forward-looking structural model. To model transparency about the averaging window, we assume that agents know to include $L-1$ lags of inflation into their PLM (but otherwise the parameters of the policy are unknown). Expectations, $\{\pi^e_{t+j}, y^e_{t+j}, R^e_{t+j}\}_{j\geq 0}$, are formed under adaptive learning.

Following the literature on adaptive learning, it is assumed that each agent has a model for perceived dynamics of state variables, also called the *perceived law of motion (PLM)*. In any period the PLM parameters are estimated using available data and the estimated model is used for forecasting. The PLM parameters are re-estimated when new data becomes available in the next period. In linearized models, a common formulation is to postulate that the PLM is a linear regression model where endogenous variables depend on intercepts, observed exogenous variables and (possibly) lags of endogenous variables.[17] The estimation is based on least squares or related methods.[18] In each period the estimated

[17]The assumption of a linear PLM is often used as an approximation also in nonlinear models as the true nonlinear functional form of the model would involve expectations of complicated nonlinear functions.

[18]For discussions of adaptive learning, see e.g. Evans and Honkapohja (2001), Sargent (2008) and Evans and Honkapohja (2009a).

PLM is substituted into the expectations in the structural model. This yields the temporary equilibrium for the period, also called *the actual law of motion (ALM)*.

We now formally introduce the general multivariate framework. Adopting the vector notation $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t,)^T$ and $\tilde{X}_t = (\hat{X}_t, ..., \hat{X}_{t-(L-1)}),^T$ the general framework is a linearized multivariate model in which agents are forward-looking with infinite horizon and there are $L-1$ lags of endogenous variables. Its general form is

$$\begin{aligned}\tilde{X}_t &= (X_t, ..., X_{t-(L-2)})^T \\ X_t &= K + \sum\nolimits_{i=1}^{\infty} \beta^i M X^e_{t,t+i} + \sum\nolimits_{j=1}^{L-1} N_j X_{t-j}.\end{aligned}$$

Stacking the system into first order form gives the temporary equilibrium system of equation

$$\tilde{X}_t = \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M} \tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1},$$

which is written out

$$\begin{aligned}\begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-(L-2)} \end{pmatrix} &= \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum\nolimits_{i=1}^{\infty} \begin{pmatrix} \beta^i M & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X^e_{t,t+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} N_1 & N_2 & \cdots & N_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ \vdots \\ X_{t-(L-1)} \end{pmatrix},\end{aligned}$$

and so

$$\begin{aligned}\tilde{X}_t &= \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-(L-2)} \end{pmatrix}, \tilde{M} = \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \tilde{X}^e_{t,t+i} &= \begin{pmatrix} X^e_{t,t+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tilde{N} = \begin{pmatrix} N_1 & N_2 & \cdots & N_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix}\end{aligned}$$

The PLM is

$$X_t = A_0 + \sum\nolimits_{j=1}^{L-1} A_j X_{t-j}$$

so

$$\tilde{X}_t = \tilde{A}_0 + \tilde{A}\tilde{X}_{t-1}, \tag{21}$$

where

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \tilde{A}_0 = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Under learning with transparency, agents' forecasting model, or (PLM) has the same $VAR(L-1)$ form as the linearized economy:[19]

$$\tilde{X}_t = \tilde{A}_0 + \tilde{A}\tilde{X}_{t-1}. \tag{22}$$

We first assume that agents know the structural form of the model, but not its parameter values.

Definition *Agents are* ***learning with transparency*** *if their linear PLM includes all L-1 lags of inflation.*

Nevertheless, agents are assumed to have imperfect knowledge about the economy's structure, including coefficients of the policy rule such as ψ_p, so they estimate the coefficients of their PLM in real-time.

The temporary equilibrium $\hat{X}_t$ is given by (18), (19), and (20) given expectations $\hat{X}^e_{t+i}$. Using the general formulation developed above, the *actual law of motion* (ALM) for the economy is

$$\tilde{X}_t = \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M}\tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1} \tag{23}$$

and, after substituting in the PLM (22), the temporary equilibrium mapping $PLM \to ALM$ can be simplified to

$$\begin{aligned} \tilde{A} &\to \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M}\tilde{A}^{i+1} + \tilde{N}, \\ \tilde{A}_0 &\to \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M}(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0. \end{aligned} \tag{24}$$

This mapping determines the E-stability conditions which are given in Appendix A.6, equations (70).[20] For brevity, in the MSV setting we only check that these conditions are satisfied in a calibrated version of the model. For the calibration see Section 3.2.

We have the results:

Result 1: *If agents are learning with transparency about the structure of the policy rule, then the target steady state is E-stable when prices are sticky.*

The flexible-price model is discussed in Appendix A.7 and this case is covered in the following:

Result 2: *The case of flexible prices is a limiting case of the NK model when the price stickiness parameter $\gamma \to 0$. The target steady state is E-stable when the policy structure is known to private agents.*

3 Imperfect Structural Knowledge and Learning

Recalling that the Federal Reserve has given little information about the details of AIT, we now start to consider situations where the policy structure, including the averaging length L is not known to the agents. Under full opacity about the monetary policy framework, agents must forecast the interest rate as well as output and inflation rate without any

[19] More precisely, agents' PLM has the same $VAR(L-1)$ form as the minimal state variable (MSV) solution of the linearized model.

[20] E-stability conditions are usually the key conditions for convergence of adaptive learning in many macroeconomic models, see Appendix A.5 for more detail and some references.

knowledge of the variables and lags of inflation in the policy rule (e.g. L). In this situation agents' learning is about how to forecast future inflation, output and interest rate and agents are assumed to exclude lagged endogenous variables from their PLM. Below we also consider the cases where agents' PLM incorporates a smaller averaging window than L. It turns out that the stability properties are the same for these intermediate cases.

Definition *Agents are* ***learning with full opacity*** *if they exclude all lagged endogenous variables (e.g. inflation) from their linear PLM, but possibly include other observables (e.g.* $\tilde{g}_t$*).*

Note that the only lags in the model are lagged inflation rates in the policy rule and private agents have no knowledge of the form (17). Consequently, it is reasonable to think they would exclude these variables from their forecasting models. Also note, just as in section 1.1, that agents' PLM under full opacity can be the model-consistent PLM in the model with a simple IT rule ($L = 1$), and therefore under full opacity agents forecast as if they are living under a standard IT regime. With AIT under full opacity, the equilibrium involves an under-parameterized forecasting model and thus the possible long-term outcome is a *restricted perceptions equilibrium*.[21]

Our interest is the stability of the model's target steady state, which can be validly assessed under the simplifying assumption that the random part of government spending $\tilde{g}_t$ is identically zero.[22] This assumption distills the full opacity PLM down to an intercept term (i.e. agents estimate the long-run mean values of state variables). We assume agents estimate these long-run mean values using a steady state learning scheme which is formalized as

$$s^e_{t+j} = s^e_t \text{ for all } j \geq 1, \text{ and } s^e_t = s^e_{t-1} + \omega_t(s_{t-1} - s^e_{t-1}), \tag{25}$$

where $s = y$, π, R. It should also be noted that in this notation expectations s^e_t refer to future periods (and not the current one) formed in period t. When forming s^e_t the newest available data point is s_{t-1}, i.e. expectations are formed in the beginning of the current period. 'Constant gain' learning is assumed, so that the gain parameter is $\omega_t = \omega$, for $0 < \omega \leq 1$ and assumed to be small.[23]

3.1 Temporary Equilibrium and Full Opacity

Under steady state learning with full opacity, agents form expectations $\pi^e_{t+j} = \pi^e_t$, $y^e_{t+j} = y^e_t$ and $R^e_{t+j-1} = R^e_t$ for $j = 1, 2, ...$ according to (25) at the beginning of time t. Time-t temporary equilibrium is given by
(i) the infinite horizon Phillips curve,

$$\begin{aligned}
\pi_t(\pi_t - 1) &= \tilde{K}(y_t, y^e_t) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu - 1}{\gamma} \frac{y_t}{(y_t - \bar{g})^\sigma} + \\
&\quad \frac{\nu}{\gamma} \sum_{j=1}^{\infty} \alpha^{-1} \beta^j \left(y^e_t\right)^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu - 1}{\gamma} \sum_{j=1}^{\infty} \beta^j \frac{y^e_t}{(y^e_t - \bar{g})^\sigma} \text{ or} \\
\pi_t &= \Pi(y_t, y^e_t) \equiv Q_{-1}[\tilde{K}(y_t, y^e_t)],
\end{aligned} \tag{26}$$

[21] See e.g. Evans and Honkapohja (2001) and Branch (2006). The term self-confirming equilibrium is also used in in the literature, see e.g. Sargent (1999).

[22] See Appendix A.5 for more details on the learning formulation.

[23] Note that the in the simple model of section 1.1 agents updated their inflation expectations using a steady state learning scheme with constant gain (see (4)).

(ii) the aggregate demand function coupled with market clearing,

$$\begin{aligned} y_t &= Y(y_t^e, \pi_t^e, R_t, R_t^e) \\ &\equiv \bar{g} + \left[\left(\frac{\beta R_t}{\pi_t^e} \right)^{-\sigma^{-1}} \left(1 - \beta^{\sigma^{-1}} \left(\frac{R_t^e}{\pi_t^e} \right)^{(1-\sigma)\sigma^{-1}} \right) \right] (y_t^e - \bar{g}) \left(\frac{R_t^e}{R_t^e - \pi_t^e} \right) \end{aligned} \quad (27)$$

(iii) and the interest rate rule (17) including the definition of inflation.

The system is compactly written

$$\begin{aligned} y_t - Y(y_t^e, \pi_t^e, R_t, R_t^e) &= 0, \\ \pi_t - \Pi(y_t, y_t^e) &= 0, \\ R_t - R(y_t, \pi_t, ..., \pi_{t+1-L}) &= 0, \end{aligned}$$

or,

$$F(X_t, X_t^e, X_{t-1}, ..., X_{t-(L-1)}) = 0, \quad (28)$$

where F consists of the aggregate demand function, the Phillips curve and the interest rate rule. The vector of current state variables is $X_t = (y_t, \pi_t, R_t)^T$ while $(X_{t-1}, ..., X_{t+1-L})^T$ contains the lagged endogenous variables. The rule of steady state learning for the components of X_t can be written in vector form as

$$X_t^e = (1-\omega)X_{t-1}^e + \omega X_{t-1}. \quad (29)$$

It should be noted that the model has two steady states, (i) the target steady state $(\pi^*, y^*, \bar{R})$, where $\bar{R} = \beta^{-1}\pi^*$ and $\pi^* = \Pi(y^*, y^*)$ and $y^* = Y(y^*, \pi^*, \bar{R}, \bar{R})$, and (ii) $(\pi_{Low}, y_{Low}, 1)$, where $\pi_{Low} = \beta^{-1}$ and $y_{Low} = Y(y_{Low}, \beta^{-1}, 1, 1)$.[24]

3.2 The Case of Full Opacity

This section considers the local stability of the target steady state under learning when agents have no knowledge of the form of the interest rate rule. We do local stability analysis, so we linearize the system (28) and (29) around the target steady state:

$$\hat{X}_t = (1-\omega)M\hat{X}_{t-1}^e + (\omega M + N_1)\hat{X}_{t-1} + \sum_{i=2}^{L-1} N_i \hat{X}_{t-i} \quad (30)$$

$$\hat{X}_t^e = (1-\omega)\hat{X}_{t-1}^e + \omega \hat{X}_{t-1}., \text{ where} \quad (31)$$

Here $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t,)^T$ with the hat denoting a linearized variable and the matrices M, N_1, ..., N_{L-1} are given in the Appendix D. 1. Recall also that X_t^e refers to expected future values of X_t and not the current one.

First, we focus on "small gain" results, i.e. whether stability obtains for all ω sufficiently close to zero.

Definition *The steady state is said to be* ***expectationally stable*** *or* ***(locally) stable under learning*** *if it is a locally stable fixed point of the system (28) and (29) for all* $0 \leq \omega < \bar{\omega}$ *for some* $\bar{\omega} > 0$.

[24]The multiplicity of steady states in New Keynesian Models is well-known. See Honkapohja and Mitra (2020) for the cases of IT and PLT. References to the literature about the multiplicity can be found e.g. in Evans et al. (2020).

Conditions for this can be directly obtained by analyzing (30)-(31) in a standard way as a system of linear difference equations.[25] Intuitively, local instability means that an arbitrarily small disturbance to agents' expectations causes the economy to permanently diverge away from the steady state. Local instability is therefore a serious warning signal about the performance of monetary policy.

As in section 1.1, we find there is local stability of constant gain learning with full opacity if there is price stickiness and the Taylor Principle is satisfied:[26]

Proposition 1 *Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $\nu > 1$ and $\sigma \geq 1$. For small ω, the target steady state is locally stable under constant gain learning with full opacity under the rule (17) for all L.*

Also as in section 1.1, we have local instability when there is full price flexibility.

Proposition 2 *Assume that there is full price flexibility ($\gamma = 0$) and $\psi_p > \beta^{-1}\pi^*$. For small ω, the target steady state is locally stable under constant gain learning with full opacity under the rule (17) for $L \leq 3$ but is unstable for higher values of L.*

Propositions 1 and 2 raise questions about applicability of the results. As stability is overturned in the limit $\gamma \to 0$ to price flexibility, it is imperative to study whether the AIT rule ensures a stable equilibrium for empirically plausible values of the gain parameter when there are positive adjustment costs $\gamma > 0$.[27] There is no agreed upon range for gain parameters but the range could be something like $[0.002, 0.04]$,[28] and so we propose the following relatively conservative definition of robust stability.

Definition *The steady state is said to be* ***robustly stable under learning*** *if it is a locally stable fixed point of the system (28) and (29) for all $0 < \omega \leq 0.01$.*

We need to calibrate the system (30) and (31) in order to assess robust stability when $\gamma > 0$. In the literature suggested calibrations for price adjustment costs, γ, vary a great deal as they depend on estimates of frequency of price adjustment and markup and there are different estimates for both. For recent discussion see Honkapohja and Mitra (2020) who use the alternative values $\gamma = 42$, 128.21 or 350 for price adjustment cost parameter. For brevity, only a single standard calibration is adopted for other parameters of our quarterly framework: $\pi^* = 1.005$, $\beta = 0.99$, $\alpha = 0.7$, $\nu = 21$, $\sigma = \varepsilon = 1$, and $g = 0.2$. Policy parameters for the AIT rule are set at $\psi_p = 1.2$, $\psi_y = 1$. We choose a low value of ψ_p and a high value of ψ_y because they bias the model in favor of stability (Appendix C.1 shows that stability outcomes deteriorate when ψ_p is high or ψ_y is low).

For the three calibrations of γ we compute the (approximate) least upper bound for the gain parameter ω_0, so that values $\omega > \omega_0$ lead to instability of the target steady state in the calibrated model. We repeat this analysis for other values of the price and output reaction coefficients in the AIT rule (17). The basic result is:

---

[25] Alternatively, so-called expectational stability (E-stability) techniques based on an associated differential equation in virtual time can be applied, e.g. see Evans and Honkapohja (2001).

[26] Proofs are given in the Appendices D.1 and D.2.

[27] The idea of using the range of values for gain parameter as a criterion for robustness was first suggested in Evans and Honkapohja (2009b).

[28] See e.g. Orphanides and Williams (2005), Branch and Evans (2006), Milani (2007) and Eusepi, Giannoni and Preston (2018).

Remark 3 *The calibrated model with sticky prices is not robustly stable for higher values of* L.

Table I demonstrates that while the IT rule is robustly stable, the AIT and PLT rules are not robustly stable. The latter two are fairly similar in terms of robustness if L is not too high (e.g. $L = 6$ is in the range of optimal averaging windows in Amano et al. (2020)).[29] However, for higher values of L such as $L = 20$ or $L = 32$, which corresponds to 5-year and 8-year averages, respectively, we observe instability for values of ω that are implausibly low.

Table I: Least upper bounds in AIT, IT and PLT

γ	42	128.21	350
ω_0 (IT)	0.02935	0.03679	0.04172
ω_0 (PLT)	0.00805	0.00590	0.00413
ω_0 (AIT with $L = 6$)	0.00404	0.00581	0.00809
ω_0 (AIT with $L = 20$)	0.00039	0.00085	0.00148
ω_0 (AIT with $L = 32$)	0.00015	0.00036	0.00070

Looking at the results so far it is apparent that a fully opaque average inflation targeting framework can pose serious risks to economic stability. In principle, communication about the averaging window may mitigate concerns about robust stability, as a transparent AIT framework does not pose the same stability risks.[30]

3.3 Opacity AR(1) case

A natural alternative to the full opacity PLM (which excludes all lags of inflation) is the following PLM:

$$s^e_{t+j} = a_{s,t} + b_{s,t}\pi^e_{t+j-1}, \tag{32}$$

where $s = y, \pi, R$. The PLM (32) is similar to PLMs studied in Hommes and Zhu (2014), and it encodes agents' belief that inflation is serially correlated. As noted previously, we say that agents have "opacity" but not "full opacity" if they include at least one lag of inflation in their PLM. We start with the $AR(1)$ case and, in the next section, discuss other underparameterized forecasting models with more than one but less than $L - 1$ lags.

Definition *Agents are* ***learning with opacity*** *if they include at least one but less than* $L - 1$ *lags of inflation in their linear PLM.*

As with full opacity, the PLM (32) is under-parameterized relative to the MSV REE law of motion, and therefore agents cannot learn an REE under opacity. However, agents' beliefs may nonetheless converge to a self-confirming restricted perceptions equilibria (RPE), (a_s, b_s), where (a_s, b_s) satisfy the following least squares orthogonality restriction:

$$E\pi_{t-1}(s_t - a_s - b_s\pi_{t-1}) = E(\pi_{t-1} - a_s)(s_t - a_s - b_s\pi_{t-1}) = 0, \tag{33}$$

[29]It may be recalled that according to Honkapohja and Mitra (2020) performance of PLT is much improved if private agents use more information about the policy framework.

[30]Appendix C.1 demonstrates that the calibrated model is robustly stable under learning with transparency.

if such an RPE exists. Hommes and Zhu (2014) uses a similar orthogonality restriction to identify self-confirming equilibria of univariate models with AR(1) PLMs. Intuitively, (33) ensures that agents' have self-confirming beliefs about the mean and the 1st auto-correlation in the data, and hence cannot easily detect the fact that their PLM is misspecified.

We now consider stability of learning in the $AR(1)$ case of opacity. To illustrate stability in a parsimonious fashion, we assume $b_{s,t} = b_s$ where b_s satisfies (33), and that agents estimate $a_{s,t}$ using the following constant-gain learning algorithm:[31]

$$a_{s,t} = a_{s,t-1} + \omega(s_{t-1} - b_s\pi_{t-2} - a_{s,t-1}) \tag{34}$$

Expectations, s^e_{t+j}, are formed by substituting (34) and $b_{s,t} = b_s$ into (32), and temporary equilibrium is given by (18), (19), and (20). In this setting, we say that the target steady state is stable under learning with opacity if agents learn an RPE near the target (i.e. an RPE for which $a_s = 0$ for $s = \hat{y}, \hat{\pi}, \hat{R}$). As it turns out, the target steady state is stable under learning if prices are sufficiently sticky, but is unstable under learning if prices are flexible.

Proposition 3 *(i) Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $\nu > 1$ and $\sigma \geq 1$. For small ω and high γ, the target steady state is locally stable under constant gain learning with AR(1) PLM (32) under the interest rate rule (17) for all L.*
(ii) Assume that there is full price flexibility ($\gamma = 0$) and $\psi_p > \beta^{-1}\pi^*$. For small ω, the target steady state is locally stable under constant gain learning with AR(1) PLM (32) under the rule (17) for $L \leq 3$ but is unstable for many higher values of L.

Comparing Proposition 1 (Proposition 2) to Proposition 3(i) (Proposition 3(ii)) it is evident that AIT poses similar risks to stability under both opacity and full opacity. Badly under-parameterized PLMs, which could arise due to a lack of transparency about the policy framework, or because households do not systematically condition their inflation expectations on a large number of lags of the quarterly inflation rate, can undermine the efficacy of AIT.

Of course there are many other under-parameterized PLMs one may consider, particularly if L is large. In the next section we provide results for these cases.

3.4 Other Underparameterized Lag Structures

It is now assumed that the lag structure in the PLM has $N-1$ lags on inflation , where $3 \leq N < L-1$.[32] Temporary equilibrium is as before, see (23) and Appendix A.6. In this setup the PLM is a $VAR(N-1)$ process while the resulting ALM is $VAR(L-1)$ and it is necessary to project the ALM into the subspace of $VAR(N-1)$ processes to obtain the best linear predictor in this class. This will give the RPE. The detailed analysis is in Appendix B.

[31] Note that we assume that $\tilde{g}_t$ is a small i.i.d shock to ensure a well-defined b_s, and it can be shown that $a_s = 0$ (i.e. the target steady state is a fixed point of the system). Further, we could employ the sample auto-correlation learning scheme in Hommes and Zhu (2014) to have agents estimate $b_{s,t}$, but this would likely lead more stringent stability conditions, and so it suffices for us to demonstrate instability under a model with intercept learning.

[32] If the number of lags in the PLM is $\geq L$, the PLM is overparameterized. This kind of situation is discussed in Chapter 9 of Evans and Honkapohja (2001).

In the sticky price setup it is not possible to obtain analytical results in the cases $1 < N - 1 < L - 1$, so we revert to numerical analysis. According to the numerical results, the stability result of the $AR(1)$ case in Proposition 3 part (i) generalizes in the sticky price economy:

Remark 4 *The result of Proposition 3 (i) continues to hold in the sticky price economy with underparameterized learning.*

Table II illustrates robustness of stability in intercept learning for $L = 8$, when agents use the RPE values for AR parameters in the PLM.[33] Table II gives (approximate) the least upper bounds ω_0 for instability.

Table II: Least upper bound for stability of the gain parameter in underparameterized constant gain learning ($L = 8$).

$N-1$	0 (Full Op.)	1 (Op.)	2	3	4	5	6	$L-1$
ω_0	0.00384	0.00400	0.00471	0.00614	0.00931	0.02382	0.02453	0.02783

In the flexible price case theoretical results can be obtained and the outcome is analogous to Proposition 3 part (ii), so the RPE is unstable under learning when agents' PLM is underparameterized.

Proposition 4 *In the flexible price economy with underparameterized learning there is no stationary learning-stable RPE when $N = 3, ...L - 1$.*

Non-existence of a stationary RPE makes it difficult to numerically compute a good approximation to the RPE. If the RPE values of the PLM parameters are not known, then it is hard to find initial conditions for a learning process that are in a small neighborhood of the RPE. Appendix B contains further discussion. The proof of Proposition 4 is in Appendix D.

4 Escape From the ZLB Regime With AIT?

One reason for introducing AIT policy has been its potential in providing a framework that facilitates return from the regime of very low interest rates to a normal regime with the economy operating near the inflation target equilibrium. From the outset, we note that AIT under full opacity is not a robust mechanism for escape from the ZLB; even if the economy escapes the ZLB regime under AIT with full opacity, the instability or non-robust stability of the target equilibrium implies that inflation never converges to the target. Further, when the ZLB is binding, the dynamics under AIT with full opacity are identical to the dynamics under IT, and so the results from earlier analyses which cast doubt on the efficacy of IT at the ZLB can be applied.[34] On the other hand, the target is robustly stable under learning with transparency. Can a transparent AIT rule bring the economy back to the target steady state from the ZLB regime?

[33] Appendix B provides the technical details.
[34] See Evans, Guse and Honkapohja (2008) and Benhabib et al. (2014).

We consider the issue of escaping the ZLB under AIT by using a stochastic version of the nonlinear model (28) with (17) and agents' PLM taking a form similar to (21)[35]

$$\tilde{X}_t = \tilde{A}_t\tilde{X}_{t-1} + \tilde{A}_{0,t} + \tilde{B}_t ln(\tilde{g}_t).$$

For this section, agents are assumed to learn about all parameters in their learning rule. At the end of each period (t), agents update their estimates, $\xi_t = (\tilde{A}_{0,t}, \tilde{A}_t, \tilde{B}_t)^T$, using all data available at the end of the period using a standard constant gain learning algorithm:

$$\begin{aligned} \xi_t &= \xi_{t-1} + \omega R_t^{-1} z_t \left(\tilde{X}_t - \xi_{t-1}^T z_t\right)^T, \\ R_t &= R_{t-1} + \omega(z_t z_t' - R_{t-1}), \end{aligned}$$

where $z_t = (1, \tilde{X}_{t-1}^T, \tilde{g}_t^T)^T$.[36] Agents make forecasts using $\tilde{A}_{0,t}, \tilde{A}_t, \tilde{B}_t$ in period $t+1$. Agents are assumed to understand that they live in the nonlinear model, and so $x_{t+j}^e = exp(\hat{x}_{t,t+j}^e)$, where $x = y, R, \pi$, hatted variables are the logs of variables, and x_{t+j}^e is the time-t expectation of x_{t+j}.

As a first example consider the case where the economy is initially very near the low steady state (y_{Low}, π_{Low}) with binding ZLB, such that $\pi_0^e = \pi(0) = \pi(-1) = \ldots = \pi(-L+1) \approx \pi_{Low}$, $y_0^e = y_0 \approx y_{Low}$, $R_0 = R_0^e \approx 1$, where y_0^e, π_0^e, and R_0^e denote the initial expected *long-run* levels of y, π, and R, respectively.[37] We assume $\tilde{A}_0$ is a zero matrix and $\tilde{A}_{0,0}$ is set in accordance with R_0^e, π_0^e, and y_0^e. Our assumption about $\tilde{A}_0$ may be a reasonable description of beliefs at the beginning of a transition from a standard IT policy regime to a well-communicated AIT regime. The basic calibration is the same as earlier in Section 3 with $\gamma = 128.21$, $\omega = 0.005$, and $L = 6$.[38] The economy escapes the liquidity trap in this case, as shown by the blue curves in Figure II.

Figure II A-C HERE

AIT with learning under transparency is also compared with the IT policy framework in Figure II. From the figure, we see that AIT generates makeup inflation and brings inflation to the level of the target much faster than under IT, though the makeup inflation comes at the expense of greater output volatility. In both cases, the economy converges to the target steady state, but this finding is not robust; when we vary L and ω, and repeat the same simulation from the low steady state, we observe convergence to target under IT for much higher values of the gain parameter than under a transparent AIT regime (see Table III). Deflationary spirals take place in simulations that do not converge to the target steady state.

Table III: Least upper bounds ω_0 for instability

L	1 (IT)	4	6	12	20
ω_0	0.06495	0.02178	0.01262	0.00582	0.00349

[35]In these simulations we include a small shock (i.e. $\tilde{g}_t \neq 0$) to avoid multicollinearity issues which arise when agents jointly estimate the intercept and lagged variable coefficients in the non-stochastic version of the model. See e.g. Evans and Honkapohja (2001), Chapter 7 for a review of related issues. The basic results of this section do not depend on the values of the shock.

[36]In simulations, agents are assumed to know that lags of output and the interest rate do not matter in equilibrium. Suitable modifications to the learning algorithm are made to impose this assumption.

[37]In fact to facilitate the numerics we set R_0 and R slightly above 1.

[38]Section 3 and Appendix C.1 analysis indicates that lower values of L may lead to better stability outcomes than higher values. We choose a value of L that is the range of optimal averaging windows studied by Amano et al. (2020).

Whether AIT initiates escape from the liquidity trap depends also on assumptions about initial expectations and economic conditions. The **domain of escape**[39] from the liquidity trap for different initial conditions $\pi_0^e \approx \pi(0) = \pi(-1) = \ldots = \pi(-L+1)$, $y_0^e \approx y_0$, and $R_0 = R_0^e \approx 1$, with $L = 6$ and low gain parameter, $\omega = .002$, is shown below in Figure III.

Figure III HERE

It is seen that there is a domain of escape from the liquidity trap, but it covers only a small area around the low steady state. In particular, if y_0^e is below y_{Low} and π_0^e is approximately at level π_{Low}, the economy does not escape from the liquidity trap.[40] By this measure both AIT and IT are less robust than PLT under similar information settings (see Honkapohja and Mitra (2020) for the corresponding results under PLT, and Appendix C for the domain of escape under IT).

We did not find the basic results in this section to be sensitive to initial values of the lag coefficients of the PLM parameters. Figure A.2 in the appendix presents the domain of escape when agents' initial beliefs about the lag coefficients, $\tilde{A}$, coincide with the coefficients of the unique, stable MSV solution of the linearized model (i.e. we impose $\tilde{A}_0 = \bar{A}$, where $\bar{A}$ is from the model linearized around the target steady state). Under this assumption about initial beliefs, agents understand that the policymaker aims for makeup inflation following a ZLB event, and yet still the performance of AIT is not significantly improved.[41]

Our analysis shows that the performance of AIT policy in the nonlinear model with the ZLB is sensitive to the speed of learning, just as the success of AIT near the target steady state hinges on the magnitude of the gain parameter. Further, AIT does not clearly outperform IT when expectations are near the low steady state.

5 Asymmetric AIT

The previous sections focus on rules that respond symmetrically to positive and negative deviations of average inflation from the target. It was shown that the central bank may have to commit to a credible, transparent averaging window or risk losing control of inflation under a symmetric rule. Such commitment may be undesirable to the policymaker depending on their preferences. However, alternative asymmetric rules have been proposed in the literature.[42]

It is beyond the scope of this paper to study asymmetric makeup strategies in great detail, but we briefly consider performance of an asymmetric AIT rule of the form

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p[\mathcal{P}_t - 1] + \psi_y[\frac{y_t - y^*}{y^*}], 0], \text{ where} \tag{35}$$

$$\mathcal{P}_t = \begin{cases} \prod_{i=0}^{L-1} \frac{\pi_{t-i}}{\pi^*} & \text{if } \prod_{i=1}^{L} \pi_{t-i} < (\underline{\pi})^L, \\ \frac{\pi_t}{\pi^*} & \text{if } \prod_{i=1}^{L} \pi_{t-i} \geq (\underline{\pi})^L, \end{cases} \tag{36}$$

[39]Domain of escape from the liquidity trap is the set of initial conditions near the low steady state that lead to convergence to target steady state. It is part of domain of attraction of (π^*, y^*).

[40]The figure also includes a line indicating the boundary of the ZLB region.

[41]Additional sensitivity analysis is available on request.

[42]See the introduction for papers that explore "switching" rules at the ZLB. The asymmetric rule we study here responds to an accumulated shortfall of inflation (over a finite horizon), and therefore resembles features of the temporary price level targeting rules and threshold rules studied in, e.g., Bernanke, Kiley and Roberts (2019).

which targets average inflation (i.e. the product of gross inflation over the last L periods) when average inflation is less than $(\underline{\pi})^L$ and targets current inflation otherwise. Note that under the asymmetric rule the policymaker aims to overshoot the target, but does not aim to undershoot the target.

If $\underline{\pi} < \pi^*$, then the asymmetric rule is a simple IT rule when average inflation is sufficiently near the level of the target. Earlier analysis shows that the target is robustly stable under a simple IT Taylor rule, which leads to the following remark.

Remark 5 *If $\underline{\pi} < \pi^*$ with $\underline{\pi}$ is sufficiently low and $\psi_p > \beta^{-1}\pi^*$ then the target steady state of the model (26) and (27), paired with asymmetric rule (35)-(36) is robustly stable under learning with full opacity for all L and $\gamma \geq 0$.*

On the other hand, the asymmetric rule coincides with the symmetric rule (17) near the target steady state if $\underline{\pi} > \pi^*$, which implies greater possibility of instability near the target equilibrium.

Remark 5 predicts that a fully opaque asymmetric rule will do a better job of bringing the economy out of the ZLB regime and back to the target steady state than a fully opaque symmetric rule. We test this prediction in the full non-linear model (26) and (27), assuming agents update beliefs using (29). As before, we assume the economy is initially near the deflation steady state (i.e. $\pi_0^e \approx \pi_0 = \pi_{-1} = ... = \pi_{-L+1} \approx \pi_{Low}$, $y_0^e \approx y_0 \approx y_{Low}$, $R_0^e \approx R_0 \approx 1$), and we set $L = 6$, $\omega = 0.015$, and $\pi_{Low} < \underline{\pi} = .995 < \pi^*$. All other calibration details are the same as before, and later we discuss robustness.

Figure IV A-C HERE

Figure IV displays results for the asymmetric rule under full opacity (blue lines). For comparison the results under a symmetric IT rule (red lines) and the benchmark symmetric AIT rule (17) under full opacity (orange lines) are also shown in the figure. Under both the asymmetric and symmetric AIT rules, makeup inflation with overshooting is observed after an initial period of low inflation, but dynamics under the rules differ especially when the downward adjustment starts. Under asymmetric AIT, overshooting of makeup inflation is very moderate and inflation does not undershoot the inflation target and, together with the interest rate, inflation gradually falls back to the target steady state. With the symmetric rule average inflation strongly overshoots the target and the policymaker abruptly raises the interest rate which makes both inflation and the interest rate undershoot the target. Since the imperfectly informed agents do not understand enough about the structure of policy to forecast this pattern of over- and undershooting, their long-run inflation expectations fall during the subsequent undershooting until deflationary spirals take hold in a second, final ZLB event.

Table IV confirms that the asymmetric rule with $\underline{\pi} < \pi^*$ outperforms the symmetric rule (as well as the asymmetric rule with $\underline{\pi} > \pi^*$) in a deep liquidity trap across a range of ω and L calibrations and for values of $\underline{\pi}$ near π^*. In Table IV, ω_0 is the largest value of ω for which we observe convergence to target steady state in simulations of the nonlinear model where the economy is initially near the low steady state. If $\underline{\pi} < \pi^*$ then ω_0 is high and similar across the IT and asymmetric AIT specifications, whereas ω_0 is low and similar across the asymmetric and symmetric AIT specifications when $\underline{\pi} > \pi^*$.

Table IV: Least upper bounds ω_0 for instability

L	1 (IT)	4	6	12	20
ω_0 ($\underline{\pi} = 1.004$)	0.06495	0.06495	0.06495	0.06495	0.06495
ω_0 ($\underline{\pi} = 1.075$)	0.06495	0.01804	0.00950	0.00312	0.00129

These results suggest that full opacity is no longer a concern if the policymaker uses an asymmetric rule of the form (35)-(36) that aims for overshooting, but not for undershooting of the target. Thus, while Hoffman et al. (2022) find that German households understand the implications of an asymmetric AIT strategy, our results suggest that asymmetric strategies can still perform well if expectations are not responsive to announcements about AIT, as Coibion et al. (2020) find in the case of U.S. households. Our findings should encourage additional research on the performance of asymmetric makeup policy rules under adaptive learning.

6 Variations on a Theme: Weighted Averages

We now consider whether the use of weighted measures of average inflation that discount past inflation relative to current inflation in computing AIT can improve stability properties of AIT policy. We consider two natural deviations from the finite, simple moving averaging schemes studied above.

6.1 Exponentially Declining Weights

First, we introduce exponentially declining weights over the finite past horizon when computing average inflation for the interest rate rule. Thus the rule (17) is adjusted to

$$R_t \equiv 1 + \max[\bar{R} - 1 + \psi_p \left[\sum_{i=0}^{L-1} \mu^i (\frac{\pi_{t-i}}{\pi^*} - 1)\right] + \psi_y [\frac{y_t}{y^*} - 1], 0], \tag{37}$$

where $0 < \mu < 1$. The length of the past horizon is $L - 1$ as before.[43] The framework is otherwise unchanged: the aggregate demand function (27), the Phillips curve (26) and learning with full opacity. The economy is stable under learning with full opacity and the rule (37) even when there is full price flexibility.

Proposition 5 *(i) Assume that there is full price flexibility ($\gamma = 0$) and $\psi_p > \beta^{-1}\pi^*$ and $0 < \mu < 1$. For all sufficiently small ω, the target steady state is locally stable under constant gain learning under the rule (37) for all L.*
(ii) Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $\nu > 1$, $\sigma \geq 1$, and $0 < \mu < 1$. For small ω, the target steady state is locally stable under constant gain learning under the rule (37) for all L.

Robustness of stability in Proposition 5 (ii) is examined using the calibrated model discussed in Section 3. Table V repeats the analysis in Table I for different values of the discount parameter μ and under the assumption that $L = 6$.

[43] Note that weights will sum to 1 if we replace ψ_p with $\psi_p \frac{1-\mu}{1-\mu^L}$. As before, we absorb the averaging constant, $\frac{1-\mu}{1-\mu^L}$, into the inflation reaction coefficient, ψ_p.

Table V: Least upper bounds ω_0 for instability

γ	42	128.21	350
ω_0 $(\mu = 1)$	0.00404	0.00581	0.00809
ω_0 $(\mu = .9)$	0.00629	0.00758	0.01059
ω_0 $(\mu = .7)$	0.01209	0.01380	0.02154
ω_0 $(\mu = .5)$	0.02312	0.02928	0.04007

It is seen that discounting old data in the AIT rule contributes robustness of stability but a significant degree of discounting is needed. We conclude that this specification only modestly improves stability outcomes.

6.2 Exponential Moving Average Rule

A different way to discount old data is to assume that an exponential moving average specification is used in the interest rate rule. Consider an interest rate rule that responds to an exponential moving average of inflation:[44]

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p \left(\frac{\pi_t^{w_c} (\pi_t^{cb})^{1-w_c}}{\pi^*} - 1 \right), 0] \quad (38)$$

$$\pi_t^{cb} = \pi_{t-1}^{w_c} (\pi_{t-1}^{cb})^{1-w_c} \quad (39)$$

where $0 < w_c < 1$. The framework is otherwise unchanged: the aggregate demand function (27), the Phillips curve (26) and steady state learning.

The dynamic model is now given by

$$F(X_t, X_t^e, \pi_t^{cb}) = 0, \quad (40)$$

where F consists of the Phillips curve, the aggregate demand function and interest rate rule (38). The vector of current state variables is $X_t = (y_t, \pi_t, R_t)^T$. The law of motion for X_t^e is the same as before, and the law of motion for π_t^{cb} is given by (39). Linearizing around the target steady state we obtain the system

$$\hat{X}_t = (-DF_X)^{-1} (DF_{X^e} \hat{X}_t^e + DF_{cb} \hat{\pi}_t^{cb}) \quad (41)$$

$$\equiv M \hat{X}_t^e + N \hat{\pi}_t^{cb} \quad (42)$$

where M and N are given in the appendix, and $\hat{X}$ again collects linearized y, π, R. In a model with sticky prices and an exponential moving average rule, the Taylor Principle is now sufficient for stability under constant gain learning:

Proposition 6 *(i) Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $0 < w_c < 1$, $\nu > 1$, $\sigma \geq 1$. For small ω, the target steady state is locally stable under constant gain learning under the rule (38).*
(ii) Assume full price flexibility ($\gamma = 0$) and $\psi_p > \max[\frac{\pi^*(\frac{\omega}{w_c})(1-w_c)}{(1-\beta)\beta}, \beta^{-1}\pi^*]$. For small ω, the target steady state is locally stable under constant gain learning under the rule (38).

[44] Budianto et al. (2023) study AIT with exponential MA in a model with bounded rationality.

When there is full price flexibility, however, the stability conditions depend on w_c and the ratio of ω to w_c, and the situation can ultimately be more stringent than the preceding proposition indicates. If ω and w_c are both relatively small (i.e. the averaging window is relatively long) and $\omega \approx w_c$, then the condition for stability is far more demanding than the Taylor Principle. Eusepi and Preston (2018), section 4 study a related model that assumes $\omega = w_c$ and obtain similar results.

The fact that stability may depend on the private sector gain parameter suggests that the exponential moving average formulation of average inflation targeting can be a risky alternative to the weighted average formulation discussed in Section 6, and future work should examine the implications of these risks.

7 Conclusion

Recent monetary policy challenges sparked interest in alternative policy frameworks, including AIT. The Federal Reserve adopted an AIT framework in 2020, but it did not communicate details about the structure of the policy, including the averaging window for the policy. This paper explored some implications of imperfect knowledge in an average inflation targeting regime with significant history dependence.

Our results suggest that policymakers should be cautious when implementing AIT. An AIT policy practiced under opacity of its details can fail to anchor expectations around the target steady state if prices are flexible or the speed of learning is anything but very slow. Moreover, an AIT policy practiced under opacity will typically fail to instigate an escape from a liquidity trap.

AIT is, however, more robust if agents know that the current policy stance depends on a specific number of lags of the inflation rate. If agents incorporate this information about the history-dependence of policy into learning, then the target steady state is fairly robustly stable, and AIT can even succeed in guiding the economy out of a liquidity trap. Furthermore, asymmetric strategies that aim for overshooting of the inflation target but not undershooting of the target, can stabilize expectations under learning with opacity about the policy rule and averaging window.

There is plenty of room for future research. As a starting point for our analysis, we assume AIT is either conducted under full or partial opacity, or in an environment in which agents fully incorporate knowledge of the structure of policy into learning. Future work should examine whether AIT can ensure a locally stable target steady state, or initiate an escape from the liquidity trap, when communication from the central bank is imperfectly credible.

This paper's findings also suggest that performance of asymmetric rules, including switching rules, under imperfect knowledge is an area worth further exploring. In this vein, future work should study the properties of rules proposed in Reifschneider and Williams (2000), Mertens and Williams (2019), Bernanke (2017), Bianchi, Melosi and Rottner (2021) among other papers. We focused on simple policy rules as a natural point of departure, but one could further study models with interest rate smoothing. Imperfect knowledge with learning may also have implications for optimal policy that have not yet been explored. Additionally, our analysis sheds light on the effects of an unanticipated implementation of AIT, and we have not studied anticipated transitions to AIT under conditions of imperfect knowledge.

Finally, it was shown that whether AIT succeeds in stabilizing expectations depends on price adjustment costs in the economy. For brevity, we abstracted from alternative models of price stickiness, such as the well-known Calvo (1983) model of infrequent price-setting. Future work could explore whether infrequent price-setting à la Calvo (1983) versus quadratic price adjustment costs in the spirit of Rotemberg (1982) matter for the stability of equilibrium under learning.

AALTO UNIVERSITY SCHOOL OF BUSINESS
BANK OF FINLAND

Appendices

A The NK Model

The objective for agent s is to maximize expected, discounted isoelastic cum quadratic utility subject to a standard flow budget constraint (in real terms) over the infinite horizon. The utility function for each period is standard except there is disutility from changing prices:

$$U_{t,s} = \frac{c_{t,s}^{1-\sigma_1}}{1-\sigma_1} + \frac{\chi}{1-\sigma_2}\left(\frac{M_{t-1,s}}{P_t}\right)^{1-\sigma_2} - \frac{h_{t,s}^{1+\varepsilon}}{1+\varepsilon} - \frac{\gamma}{2}\left(\frac{P_{t,s}}{P_{t-1,s}} - 1\right)^2 \quad (43)$$

and the flow budget constraint is

$$st.\ c_{t,s} + m_{t,s} + \mathfrak{b}_{t,s} + \Upsilon_{t,s} = m_{t-1,s}\pi_t^{-1} + R_{t-1}\pi_t^{-1}\mathfrak{b}_{t-1,s} + \frac{P_{t,s}}{P_t}y_{t,s}. \quad (44)$$

The final term in the utility function parameterizes the cost of adjusting prices in the spirit of Rotemberg (1982). The Rotemberg formulation is used rather than the Calvo (1983) model of price stickiness because it enables us to study global dynamics in the nonlinear system. The household decision problem is also subject to the usual "no Ponzi game" (NPG) condition. In (43) the expectations $E_{0,s}(.)$ are in general subjective and may not be rational.

Production function for good s is standard

$$y_{t,s} = h_{t,s}^{\alpha}, \text{ where } 0 < \alpha < 1. \quad (45)$$

There is no capital. Output is differentiated and firms operate under monopolistic competition. Each firm faces a downward-sloping demand curve

$$P_{t,s} = \left(\frac{y_{t,s}}{y_t}\right)^{-1/\nu} P_t. \quad (46)$$

Here $P_{t,s}$ is the profit maximizing price set by firm s consistent with its production $y_{t,s}$. The parameter ν is the elasticity of substitution between two goods and is assumed to be greater than one. y_t is aggregate output, which is exogenous to the firm.

The market clearing condition is

$$c_t + g_t = y_t.$$

The government consumes amount g_t of the aggregate good, collects the real lump-sum tax Υ_t from each consumer and issues bonds $\mathfrak{b}_t$ to cover financing needs. Fiscal policy is assumed to follow a linear tax rule for lump-sum taxes $\Upsilon_t = \kappa_0 + \kappa \mathfrak{b}_{t-1}$, where $\beta^{-1} - 1 < \kappa < 1$, so fiscal policy is "passive" using terminology of Leeper (1991). Government purchases g_t are taken to be stochastic, so that $g_t = \bar{g} + \tilde{g}_t$, where the random part $\tilde{g}_t$ is an observable exogenous AR process

$$\tilde{g}_t = \rho \tilde{g}_{t-1} + v_t \tag{47}$$

with zero mean.[45]

A.1 Private sector optimization

Using the utility function of household-producer s (43) and the budget constraint (44) and production function (45), one computes the derivatives with respect to $(t-1)$-dated variables

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t-1,s}} &= c_{t,s}^{-\sigma_1} \pi_t^{-1} + \chi (m_{t-1,s} \pi_t^{-1})^{-\sigma_2}, \\ \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t-1,s}} &= c_{t,s}^{-\sigma_1} R_{t-1} \pi_t^{-1}, \end{aligned}$$

and with respect to t-dated variables

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t,s}} &= \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t,s}} = -c_{t,s}^{-\sigma_1}, \\ \frac{\partial U_{t,s}}{\partial P_{t,s}} &= c_{t,s}^{-\sigma_1} Y_t (1-\upsilon) \left(\frac{P_{t,s}}{P_t} \right)^{-\upsilon} \frac{1}{P_t} + \frac{\upsilon}{\alpha} h_{t,s}^{1+\varepsilon} \frac{1}{P_{t,s}}. \end{aligned}$$

The Euler equations are

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial m_{t,s}} &= 0, \\ \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial \mathfrak{b}_{t,s}} &= 0, \\ \frac{\partial U_{t,s}}{\partial P_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial P_{t,s}} &= 0. \end{aligned}$$

The second equation is just the consumption Euler equation, while combining the first and second equations yields the money demand function. The third equation is the condition for optimal price setting .

Applying the above conditions, in period t each household s is assumed to maximize its anticipated utility under given expectations. As in Evans et al. (2008), the first-order conditions for an optimum yield

$$\begin{aligned} 0 \;=\; & -h_{t,s}^{\varepsilon} + \frac{\alpha\gamma}{\nu} (\pi_{t,s} - 1) \pi_{t,s} \frac{1}{h_{t,s}} \\ & + \alpha \left(1 - \frac{1}{\nu}\right) y_t^{1/\nu} \frac{y_{t,s}^{(1-1/\nu)}}{h_{t,s}} c_{t,s}^{-\sigma} - \frac{\alpha\gamma\beta}{\nu} \frac{1}{h_{t,s}} E_{t,s} (\pi_{t+1,s} - 1) \pi_{t+1,s}, \end{aligned} \tag{48}$$

$$c_{t,s}^{-\sigma} = \beta R_t E_{t,s} \left(\pi_{t+1}^{-1} c_{t+1,s}^{-\sigma} \right), \tag{49}$$

[45] For simplicity, it is assumed ρ is known (if not it could be estimated during learning). Only one shock is introduced to have a simple exposition.

where $\pi_{t+1,s} = P_{t+1,s}/P_{t,s}$ and $E_{t,s}(.)$ denotes the (not necessarily rational) expectations of agents s formed in period t.

Equation (48) is one form of the nonlinear New Keynesian Phillips curve describing the optimal price-setting by firms. The term $(\pi_{t,s} - 1)\,\pi_{t,s}$ arises from the quadratic form of the adjustment costs, and this expression is increasing in $\pi_{t,s}$ over the allowable range $\pi_{t,s} \geq 1/2$. Equation (49) is the standard Euler equation giving the intertemporal first-order condition for the consumption path.

We now write the decision rules for consumption and inflation so that they depend on forecasts of key variables over the infinite horizon (IH).

A.2 The Infinite-Horizon Phillips Curve

Starting with (48), let

$$Q_{t,s} = (\pi_{t,s} - 1)\,\pi_{t,s}. \tag{50}$$

The appropriate root for given Q is $\pi \geq \frac{1}{2}$ and so $Q \geq -\frac{1}{4}$ must be imposed to have a meaningful model. Using the production function $h_{t,s} = y_{t,s}^{1/\alpha}$ one can rewrite (48) as

$$Q_{t,s} = \frac{\nu}{\alpha\gamma} y_{t,s}^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t^{1/\nu} y_{t,s}^{(\nu-1)/\nu} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}, \tag{51}$$

and using the demand curve $y_{t,s}/y_t = (P_{t,s}/P_t)^{-\nu}$ gives

$$Q_{t,s} = \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}.$$

Defining

$$x_{t,s} \equiv \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} Y_t^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} \tag{52}$$

and iterating the Euler equation[46] yields

$$Q_{t,s} = x_{t,s} + \sum_{j=1}^{\infty} \beta^j E_{t,s} x_{t+j,s}, \tag{53}$$

provided that the transversality condition

$$\beta^j E_{t,s} x_{t+j,s} \to 0 \text{ as } j \to \infty \tag{54}$$

holds. It can be shown that (54) is an implication of the necessary transversality condition for optimal price setting. For further details see Benhabib et al. (2014).

The variable $x_{t+j,s}$ is a mixture of aggregate variables and the agent's own future decisions. Thus it provides only a "conditional decision rule".[47] This equation for $Q_{t,s}$ can be the basis for decision-making as follows. So far only the agents' price-setting Euler equation and the above limiting condition (54) have been used. Some further assumptions are now made.

Agents are assumed to have point expectations, so that their decisions depend only on the mean of their subjective forecasts. The model stipulates that all agents have the same

[46] It is assumed that expectations satisfy the law of iterated expectations.

[47] Conditional demand and supply functions are well known concepts in microeconomic theory.

utility and production functions. Initial money and debt holdings, and prices are assumed to be identical.

The assumption of representative agents includes private agents' forecasting, so that the agents have homogenous forecasts of the relevant variables. Thus all agents make the same decisions at each point in time. It is also assumed that from the past agents have learned the market clearing relation in temporary equilibrium, i.e. $c_{t,s} = y_t - g_t$ in per capita terms and thus agents impose in their forecasts that $c^e_{t+j} = y^e_{t,t+j} - g^e_{t,t+j}$, where $g^e_{t,t+j} = \bar{g} + \rho^j \tilde{g}_t$. In the case of constant fiscal policy this becomes $c^e_{t+j} = y^e_{t+j} - \bar{g}$.

The assumption of representative agents implies that $P_{t,s} = P_{t,s'} = P_t$ for all agents s and s' in temporary equilibrium for all periods including the current one, see p. 224 in Benhabib et al. (2014). In that paper it was additionally assumed that agents' expectations also satisfy $P^e_{t+j,s} = P^e_{t+j}$ for future periods $j = 1, 2, ...$ This assumption is not necessary and is adopted here purely as a simplification.[48]

A.3 The Consumption Function

To derive the consumption function from (49), use the flow budget constraint and the NPG condition to obtain an intertemporal budget constraint. Cashless limit is now assumed. First, define the asset wealth

$$\mathfrak{a}_t = \mathfrak{b}_t$$

as the holdings of real bonds and write the flow budget constraint as

$$\mathfrak{a}_t + c_t = y_t - \Upsilon_t + r_t \mathfrak{a}_{t-1}, \tag{55}$$

where $r_t = R_{t-1}/\pi_t$. Note that $(P_{jt}/P_t)y_{jt} = y_t$ is assumed, i.e. the representative agent assumption is invoked. Iterating (55) forward and imposing

$$\lim_{j\to\infty} (D^e_{t,t+j})^{-1} \mathfrak{a}^e_{t+j} = 0, \tag{56}$$

where

$$D^e_{t,t+j} = \frac{R_t}{\pi^e_{t+1}} \prod_{i=2}^{j} \frac{R^e_{t+i-1}}{\pi^e_{t+i}}$$

with $r^e_{t+i} = R^e_{t+i-1}/\pi^e_{t+i}$, one obtains the life-time budget constraint of the household

$$\begin{aligned} 0 &= r_t \mathfrak{a}_{t-1} + \Phi_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Phi^e_{t+j} && (57) \\ &= r_t \mathfrak{a}_{t-1} + \phi_t - c_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\phi^e_{t+j} - c^e_{t+j}), && (58) \end{aligned}$$

where

$$\begin{aligned} \Phi^e_{t+j} &= y^e_{t+j} - \Upsilon^e_{t+j} - c^e_{t+j}, && (59) \\ \phi^e_{t+j} &= \Phi^e_{t+j} + c^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}. \end{aligned}$$

Here all expectations are formed in period t, which is indicated in the notation for $D^e_{t,t+j}$ but is omitted from the other expectational variables.

[48] More extensive discussion of the generalization is available in Evans et al. (2020).

Invoking the relations

$$c^e_{t+j} = (\beta^j D^e_{t,t+j})^{1/\sigma} c_t, \tag{60}$$

which are an implication of the consumption Euler equation (49), yields

$$c_t(1-\beta)^{-1} = r_t \mathfrak{a}_{t-1} + y_t - \Upsilon_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j} - \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\beta^j D^e_{t,t+j})^{1/\sigma} c_t. \tag{61}$$

As we have $\phi^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}$, it follows that

$$c_t = \left(1 + \sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma}\right)^{-1} \left(r_t \mathfrak{b}_{t-1} + \sum_{j=0}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j}\right).$$

So far it is not assumed that households act in a Ricardian way, i.e. they have not imposed the intertemporal budget constraint (IBC) of the government. To simplify the analysis, it is now assumed that consumers are Ricardian, which allows to modify the consumption function as in Evans and Honkapohja (2010). See Evans, Honkapohja and Mitra (2012) for discussion of the assumptions under which Ricardian Equivalence holds along a path of temporary equilibria with learning if agents have an infinite decision horizon.

The government flow constraint is

$$\mathfrak{b}_t + \Upsilon_t = \bar{g} + \tilde{g}_t + r_t \mathfrak{b}_{t-1} \text{ or } \mathfrak{b}_t = \Delta_t + r_t \mathfrak{b}_{t-1} \text{ where } \Delta_t = \bar{g} + \tilde{g}_t - \Upsilon_t.$$

By forward substitution, and assuming

$$\lim_{T\to\infty} D_{t,t+T} = 0, \tag{62}$$

one gets

$$0 = r_t \mathfrak{b}_{t-1} + \Delta_t + \sum_{j=1}^{\infty} D^{-1}_{t,t+j} \Delta_{t+j}. \tag{63}$$

Note that Δ_{t+j} is the primary government deficit in $t+j$, measured as government purchases less lump-sum taxes. Under the Ricardian assumption, agents at each time t expect this constraint to be satisfied, i.e.

$$\begin{aligned} 0 &= r_t \mathfrak{b}_{t-1} + \Delta_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Delta^e_{t+j}, \text{ where} \\ \Delta^e_{t+j} &= \bar{g} + \rho^j \tilde{g}_t - \Upsilon^e_{t+j} \text{ for } j = 1, 2, 3, \ldots. \end{aligned}$$

A Ricardian consumer assumes that (62) holds. His flow budget constraint (55) can then be written as:

$$\mathfrak{b}_t = r_t \mathfrak{b}_{t-1} + \psi_t, \text{ where } \psi_t = y_t - \Upsilon_t - c_t.$$

The relevant transversality condition is now (62). Iterating forward and using (60) together with (62) yields the consumption function (11) in the main text.

The aggregate demand function takes the form

$$y_t = g_t + \left(\sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma}\right)^{-1} \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t)), \tag{64}$$

where the discount factor is given by (12).

A.4 Linearized IH Behavioral Rules

Linearizing (9) and (53) around the intended steady state and rearranging gives the following linearized expression for the Phillips curve:

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j}$$

where $\hat{x}$ denotes a linearized variable, and κ is a complicated function of deep structural parameters.

The consumption function (11) is linearized as follows. For the sake of brevity, assume $\tilde{g}_t = 0$. The discount factor $D^e_{t,t+j}$ has the linearization

$$\hat{D}^e_{t,t+j} = \beta^{1-j} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right).$$

Linearizing the left-hand-side of (11) gives

$$\begin{aligned} & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{j/\sigma} \left(\beta^{-j}\right)^{(1-\sigma)/\sigma - 1} \hat{D}^e_{t,t+j} \\ = & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\ = & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right). \end{aligned}$$

Linearizing the right-hand-side of (11) gives

$$\begin{aligned} & \sum_{j\geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j\geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\ = & \sum_{j\geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j\geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right). \end{aligned}$$

Equating the two sides of (11) and rearranging gives

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty} \beta^j \left(\frac{1-\beta}{\beta} \hat{y}^e_{t+j} - \frac{c^*}{\sigma} \left(\beta \hat{R}^e_{t+j}/\pi^* - \hat{\pi}^e_{t+j}/(\beta\pi^*) \right) \right). \tag{65}$$

The third equation is the linearized AIT interest rate rule (20).

A.5 Formulation of Learning

The basic model apart from the AIT rule is purely forward-looking while the observable exogenous shock $\tilde{g}_t$ is an AR(1) process. Assuming full opacity about AIT rule, then the appropriate PLM is a linear projection of $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and agents estimate the regressions

$$s_u = a_s + b_s \tilde{g}_{u-1} + \varepsilon_{s,u}, \tag{66}$$

where $s = y,\ \pi,\ R$ by using a version of least squares and data for periods $u = 1, ..., t-1$. The latter is a common timing assumption in the learning literature; at the end of period $t-1$ the parameters of (66) are estimated using data through to period $t-1$. Usually, the estimation is done using recursive least squares. This gives estimates $a_{y,t-1}$, $b_{y,t-1}$, $a_{\pi,t-1}$, $b_{\pi,t-1}$, $a_{R,t-1}$, $b_{R,t-1}$ and using these estimates and data at time t the forecasts are given by

$$s^e_{t+j} = a_{s,t-1} + b_{s,t-1}\rho^j \tilde{g}_t,$$

for future periods $t+j$. These forecasts are then substituted into the system to determine a temporary equilibrium (also called the actual law of motion (ALM) of the economy in period t. With the new data point the estimates are updated and the process continues.

Denoting the PLM parameters by $\theta_t = (a_{y,t-1},\ b_{y,t-1},\ a_{\pi,t-1},\ b_{\pi,t-1},\ a_{R,t-1},\ b_{R,t-1})$, the parameters are mapped into new values, so there is a mapping $\theta_t \to T(\ \theta_t)$, the ALM parameters. The system consisting of temporary equilibrium and estimation equations is formally a stochastic recursive algorithm (SRA) and its convergence to equilibrium depends on the properties of $T(\theta)$. It should be noted that the SRA may be written in term of decreasing or constant gain. The sense of probabilistic convergence is different in these latter two setups. Numerical analysis of this setup is done by simulating the SRA. It is possible to obtain analytical conditions for the stochastic convergence of the SRA to a fixed point θ^*. It turns out that conditions for convergence can be studied by examining the map $\theta \to T(\theta)$ and stability conditions of the ordinary differential equation $d\theta/d\tau = T(\theta)$ in virtual time τ yield convergence conditions for the real-time SRA. For example, see Evans and Honkapohja (2001) or Evans and Honkapohja (2009a) the theory and many applications.

It turns out that the technical analysis of convergence and computation of domains of attraction can be carried out using a simplification. Apart from the unknown policy rule the model is purely forward-looking while $\tilde{g}_t$ is an AR(1) process. Under full opacity the PLM is a linear projection of the state variables $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and in this case convergence of learning to a fixed point is fully governed by the dynamics of intercepts.

Thus, stability of a steady state can be validly assessed using the simplifying assumption that $\tilde{g}_t$ is identically zero. The agents are thought to estimate the long-run mean values of state variables, called "steady state learning". The latter is used here as a technical tool. In simulations of the stochastic model agents are assumed to do least squares learning.

A.6 E-Stability for Linear Multivariate IH Models

Recall the system in first vector form

$$\tilde{X}_t = \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M} \tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1}. \tag{67}$$

Consider first the MSV case where $N = L$ and the PLM is (21).The mapping $PLM \to ALM$ is (24). Assuming that the eigenvalues of $\tilde{A}$ and $\beta\tilde{A}$ are inside the unit circle, the mapping $PLM \to ALM$ simplifies to

$$\tilde{A} \ \to\ \beta\tilde{M}\tilde{A}^2\left(I-\beta\tilde{A}\right)^{-1} + \tilde{N} \tag{68}$$

$$\tilde{A}_0 \ \to\ \tilde{K} + \beta\tilde{M}\left(I-\tilde{A}\right)^{-1}\left((1-\beta)^{-1}I - \tilde{A}^2\left(I-\beta\tilde{A}\right)^{-1}\right)\tilde{A}_0. \tag{69}$$

In this case it is straight-forward to obtain the E-stability conditions.

E-stability Conditions: Let $(\tilde{A}, \tilde{A}_0) = (\bar{A}, \bar{A}_0)$ denote a rational expectations equilibrium. The REE, $(\bar{A}, \bar{A}_0)$, is E-stable if the real parts of the eigenvalues of

$$\begin{aligned} DT(\tilde{A}) &= \left(\left(I - \beta\bar{A}\right)^{-1}\beta\bar{A}^2\right)^T \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\right) + \\ & \quad I \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\bar{A}\right) + \bar{A}^T \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\right) \\ DT(\tilde{A}_0) &= \beta\tilde{M}\left(I - \tilde{A}\right)^{-1}\left((1-\beta)^{-1}I - \tilde{A}^2\left(I - \beta\tilde{A}\right)^{-1}\right) \end{aligned} \tag{70}$$

are less than one.

A.7 Model with Flexible Prices

In the special case of the NK model with flexible prices there is no Phillips curve and the first order condition (48) is replaced by the static condition

$$\frac{\partial U_{t,s}}{\partial P_{t,s}} = c_{t,s}^{-\sigma_1} Y_t (1-\upsilon)\left(\frac{P_{t,s}}{P_t}\right)^{-\upsilon}\frac{1}{P_t} + \frac{\upsilon}{\alpha} h_{t,s}^{1+\varepsilon}\frac{1}{P_{t,s}} = 0.$$

Under symmetry it yields

$$c_t^{-\sigma_1}\alpha\frac{1-\upsilon}{\upsilon} + h_t^{1+\varepsilon-\alpha} = 0, \tag{71}$$

Steady-state learning with point expectations is formalized as before in Section 3. The temporary equilibrium equations with steady state learning are as follows.

1. With Ricardian consumers the market clearing equation is $y_t = g_t + c_t$ and yields

$$y_t = \bar{g} + (1-\beta)\left[y_t - \bar{g} + (y_t^e - \bar{g})\left(\frac{\pi_t^e}{R_t}\right)\left(\frac{R_t^e}{R_t^e - \pi_t^e}\right)\right] \tag{72}$$

as the aggregate demand relation.

2. The static labor-consumption optimality condition (71) can be combined with market clearing to obtain

$$y_t = \left(\alpha\frac{\upsilon - 1}{\upsilon}(y_t - g_t)^{-\sigma_1}\right)^{\alpha/(1+\varepsilon-\alpha)}. \tag{73}$$

Looking at (73) it is evident that output in temporary equilibrium is exogenous.[49]

3. Interest rate rule (14) as discussed in the text.

If one substitutes the interest rate rule (14) and also an exogenous value of output into (72), the model effectively says that the nominal interest rate R_t (and π_t via the policy rule) is the variable that establishes equality of aggregate demand and supply in temporary equilibrium. Using the interest rate rule (14) then yields the temporary equilibrium value for inflation π_t.

The system has three expectational variables: output y_t^e, inflation π_t^e, and interest rate R_t^e. The evolution of expectations is in accordance with steady state learning. Proposition 2 gives the instability result.

[49] Exogeneity of output holds in the classical monetary model, see e.g. Gali (2008), chapter 1.

B The ALM When There is Underparameterization

B.1 General setup

The model (67) is now modified to include an underparameterized PLM

$$\begin{aligned}
\tilde{X}_{1,t} &= A_0 + \tilde{A}_N \tilde{X}_{1,t-1}, \text{ where} && (74)\\
\tilde{X}_{1,t} &= \begin{pmatrix} \hat{X}_t \\ \vdots \\ \hat{X}_{t-(N-2)} \end{pmatrix} \text{ and } \tilde{X}_t = \begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix},\\
\tilde{A}_N &= \begin{pmatrix} A_1 & \cdots & A_{N-2} & A_{N-1} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} \text{ and } \tilde{A}_0 = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned}$$

The vector $\tilde{X}_{1,t}$ is $3(N-1)$ dimensional and it consists of the current and first $N-2$ lagged endogenous variables. Thus the first block of $\tilde{X}_{1,t}$ is

$$\hat{X}_t = A_0 +_{i=1}^{N-1} A_i \hat{X}_{t-i}, \qquad (75)$$

while the other blocks express identities. Here $\tilde{A}_N$ is a square matrix with dimensions $3(N-1)$, where $N-1$ is the number of lags in the PLM of agents. Matrices A_i are 3×3, while $\tilde{A}_0$ and A_0 are $3(N-1)$ and 3 dimensional column vectors, respectively.

Iterating the PLM, we get

$$\tilde{X}^e_{1,t,t+i} = (I + \tilde{A}_N + ... + \tilde{A}_N^i)\tilde{A}_0 + \tilde{A}_N^{i+1}\tilde{X}_{1,t-1}$$

and the unprojected ALM is

$$\begin{aligned}
\begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix} &= \left(\begin{bmatrix} \tilde{K} + \bar{M}_{11} \sum_{i=1}^{\infty} \beta^i [(I + \tilde{A}_N + ... + \tilde{A}_N^i)\tilde{A}_0 + \tilde{A}_N^{i+1}\tilde{X}_{1,t-1}] \\ 0 \end{bmatrix} + \right.\\
&\left. \begin{pmatrix} \hat{N} & \hat{N} & \cdots & \hat{N} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_{t-1} \\ \hat{X}_{t-2} \\ \vdots \\ \hat{X}_{t-(L-1)} \end{pmatrix} \right),
\end{aligned}$$

where $\bar{M}_{11}$ is the submatrix of $\tilde{M}$ formed from the first $3(N-1)$ rows and columns of $\tilde{M}$ in (23). It is seen that the ALM is not in the same space as the PLM as there are further lags of $\hat{X}_t$ in the ALM. The map of the intercept term is

$$\begin{aligned}
A_0 &\to \tilde{K} + \bar{M}_{11} \sum_{i=1}^{\infty} \beta^i (I + \tilde{A}_N + ... + \tilde{A}_N^i)\tilde{A}_0 && (76)\\
&= \tilde{K} + \bar{M}_{11} \left(\sum_{i=1}^{\infty} \beta^i (I - \tilde{A}_N)^{-1} (I - \tilde{A}_N^{i+1}) \right) \tilde{A}_0\\
&= \tilde{K} + \bar{M}_{11} (I - \tilde{A}_N)^{-1} [\frac{\beta}{1-\beta} I - \beta \tilde{A}_N^2 (I - \beta \tilde{A}_N)^{-1}] \tilde{A}_0,
\end{aligned}$$

if the eigenvalues of $\tilde{A}_N$ and $\beta\tilde{A}_N$ are inside the unit circle.

Noting that

$$\bar{M}_{11}\beta\tilde{A}_N^2\sum\nolimits_{i=0}^{\infty}\beta^i\tilde{A}_N^i = [\bar{M}_{11}\beta\tilde{A}_N^2(I-\beta\tilde{A}_N)^{-1}],$$

the autoregressive term of the unprojected ALM can be written

$$\begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix} = \begin{pmatrix} \bar{M}_{11}\beta\tilde{A}_N^2(I-\beta\tilde{A}_N)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{X}_{1,t-1} \\ \tilde{X}_{2,t-1} \end{pmatrix} + \begin{pmatrix} \hat{N}(\hat{X}_{t-1}+...,+\hat{X}_{t-(L-1)}) \\ \hat{X}_{t-2} \\ \vdots \\ \hat{X}_{t-(L-1)} \end{pmatrix} + Z_t. \tag{77}$$

where Z_t is a white noise disturbance term. Define

$$F(\tilde{A}_N) = [\bar{M}_{11}\beta\tilde{A}_N^2(I-\beta\tilde{A}_N)^{-1}]$$

is $3(N-1)\times 3(N-1)$ while the dimension of the whole system is $3(L-1)\times 3(L-1)$. Note also the form of $\bar{M}_{11}$, which is zero except for top-left 3×3 corner. The (unprojected) ALM is $VAR(L-1)$ in the vector $\hat{X}_t$ while the PLM is a $VAR(N-1)$ process, and it is necessary to map the unprojected ALM into the space of $VAR(N-1)$ processes. Then compute

$$\begin{aligned} F(\tilde{A}_N) &= \begin{pmatrix} \beta M(A_1^2+A_2) & \cdots & \beta M(A_1A_{N-2}+A_{N-1}) & \beta MA_1A_{N-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &\quad\times\left(I-\beta\begin{pmatrix} A_1 & A_2 & \cdots & A_{N-2} & A_{N-1} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}\right)^{-1} \end{aligned} \tag{78}$$

Looking at this form of $F(\tilde{A}_N)$, we note that in the first matrix all except the first row blocks of 3×3 matrices are zero. This allows writing $F(\tilde{A}_N)$ in the form

$$F(\tilde{A}_N) = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_{N-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{79}$$

and the unprojected ALM is

$$\hat{X}_t^{ALM} = (\mathcal{M}_1+\hat{N})\hat{X}_{t-1}+...+(\mathcal{M}_{N-1}+\hat{N})\hat{X}_{t-(N-1)}+\hat{N}\hat{X}_{t-N}+...+\hat{N}\hat{X}_{t-(L-1)}+Z_t.$$

Next, denote the projected ALM as

$$\hat{X}_t^{PR} = \sum_{i=1}^{N-1} B_i\hat{X}_{t-i}+\varepsilon_t.$$

Then the mean forecast error is $ES = E[\hat{X}_t^{ALM} - \hat{X}_t^{PR}]$ or

$$ES = \sum_{i=1}^{N-1} (\mathcal{M}_i + \hat{N} - B_i)\hat{X}_{t-i} + \hat{N}(\hat{X}_{t-N} + ... + \hat{X}_{t-(L-1)}).$$

Minimizing the square of the mean forecast error yields the orthogonality conditions:

$$0 = E[(ES)\hat{X}_{t-j}^T] = \sum_{i=1}^{N-1} E[(\mathcal{M}_i + \hat{N} - B_i)\hat{X}_{t-i}\hat{X}_{t-j}^T + \hat{N}(\hat{X}_{t-N} + ... + \hat{X}_{t-(L-1)})\hat{X}_{t-j}^T], \text{ for } j = 1, ...N-1.$$

Now let $\hat{X}_{t-i} = \left(\hat{y}_{t-i}, \hat{R}_{t-i}, \hat{\pi}_{t-i}\right)^T$ and define[50]

$$E\hat{X}_{t-i}\hat{X}_{t-j}^T = \begin{pmatrix} E(\hat{y}_{t-i}\hat{y}_{t-j}) & E(\hat{y}_{t-i}\hat{R}_{t-j}) & E(\hat{y}_{t-i}\hat{\pi}_{t-j}) \\ E(\hat{y}_{t-j}\hat{R}_{t-i}) & E(\hat{R}_{t-i}\hat{R}_{t-j}) & E(R_{t-i}\hat{\pi}_{t-j}) \\ E(\hat{\pi}_{t-j}\hat{\pi}_{t-i}) & E(R_{t-j}\hat{\pi}_{t-i}) & E(\hat{\pi}_{t-i}\hat{\pi}_{t-j}) \end{pmatrix} = \Omega_{i.j}, \tag{80}$$

Then

$$0 = E[(ES)\hat{X}_{t-j}^T] = \sum_{i=1}^{N-1} (\mathcal{M}_i + \hat{N} - B_i)\Omega_{i,j} + \hat{N}\sum_{k=N}^{L-1} \Omega_{k,j}, \text{ for } j = 1, ...N-1. \tag{81}$$

We also need to use the Yule-Walker equations to compute $\Omega_{k,j}(m,n)$ for $k, j = 1, ..., L-1$ and $m, n = 1, 2, 3$.[51] Let Ω_{kj} denote the 3×3 matrix with elements $\Omega_{k,j}(m,n)$. Then use (79) and define

$$F_{ext}(\tilde{A}_N) = \begin{pmatrix} \mathcal{M}_1 + \hat{N} & \mathcal{M}_2 + \hat{N} & \cdots & \mathcal{M}_{N-1} + \hat{N} & \hat{N} & \cdots & \hat{N} & \hat{N} \\ I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & I & 0 \end{pmatrix}$$

which is $3(L-1) \times 3(L-1)$ matrix. Consider its eigenvalues from the system

$$\det[F_{ext}(\tilde{A}_N) - \lambda I] = 0.$$

Then define

$$\mathfrak{A} = F_{ext}(\tilde{A}_N) \otimes F_{ext}(\tilde{A}_N)$$

and obtain the matrix of second moments of the $VAR(L-1)$ process (77). in vector form

$$vec(\Sigma) = (I - \mathfrak{A})^{-1} vec(Q_Z),$$

[50] This standard procedure presumes that $\hat{X}_{t-i}$ is covariance stationary. However, the algebraic operation can be done even if stationarity is not presupposed.

[51] The method is explained well in Section 10.2 of Hamilton (1994).

where $\Sigma = (\omega_{ij}) = E(\hat{X}_t^{ALM}(\hat{X}_t^{ALM})^T)$ and Q_Z is the covariance matrix of the augmented form of the error term Z_t. Alternatively, one can use the linear equation system

$$\Sigma = F_{ext}(\tilde{A}_N)\Sigma F_{ext}(\tilde{A}_N)^T + Q_Z.$$

Introducing the notation

$$\Sigma = \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{L-2} \\ \Gamma_1^T & \Gamma_0 & \cdots & \Gamma_{L-3} \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_{L-2}^T & \Gamma_{L-3}^T & \cdots & \Gamma_0 \end{pmatrix},$$

we have $\Gamma_i = E(\hat{X}_t^{ALM}(\hat{X}_{t-i}^{ALM})^T)$ for $i = 1, ..., L-2$. The $i'th$ autocovariance matrix of the original process (77) is then

$$\Gamma_i = (\mathcal{M}_1 + \hat{N})\Gamma_{i-1} + ... + (\mathcal{M}_{N-1} + \hat{N})\Gamma_{i-(N-1)} + \hat{N}\Gamma_{i-N} + ... + \hat{N}\Gamma_{i-(L-1)}$$

for $i = L-1, L, ...$ from which the required covariances $\Omega_{kj}(m,n)$ for $k, j = 1, ..., L-1$ and $m, n = 1, 2$ are obtained and substituted into (80). As $\Omega_{ij} = E\hat{X}_{t-i}\hat{X}_{t-j}^T$ we have

$$\Gamma_{j-i} = \Omega_{ij} = \begin{cases} \Gamma_{j-i} \text{ for } j \geq i \\ \Gamma_{j-i}^T \text{ for } j < i. \end{cases}.$$

These equations are used in the numerical analysis reported in Table II and in Appendix B.3.

B.2 Stability in the sticky price case

We begin with the technical details for Remark 4 in the case of PLM lag length, $N-1$, where $1 < N \leq L-1$. For simplicity, we assume that exogenous shocks are mean-zero and i.i.d, and that agents have the following *PLM*

$$\begin{aligned} \tilde{X}_t &= \tilde{A}_N\tilde{X}_{t-1} + \tilde{A}_{0,t-1} \\ \tilde{A}_{0,t} &= \omega(\tilde{X}_t - \tilde{A}_N\tilde{X}_{t-1} - \tilde{A}_{0,t-1}) + \tilde{A}_{0,t-1}. \end{aligned}$$

$\tilde{A}_N$ contains the RPE coefficients (or the correct ALM coefficients in the case $N = L$), imposes zeros on lags of $\hat{X}$ exceeding $N-1$, and is fixed over time (i.e. agents only estimate the intercept term recursively). The actual law of motion for $\tilde{X}_t$ can be expressed as

$$\tilde{X}_t = B\tilde{X}_{t-1} + DT_a\tilde{A}_{0,t-1} + shocks$$

where B and DT_a are functions of $\tilde{A}_N$ and the model's structural parameters. This implies the following law of motion for $W_t = (\tilde{X}_t', \tilde{A}_{0,t}')'$

$$\begin{aligned} W_t &= \begin{pmatrix} B & DT_a \\ \omega(B - \tilde{A}_N) & \omega DT_a + (1-\omega)I \end{pmatrix} W_{t-1} + shocks \\ &= DT * W_{t-1} + shocks \end{aligned}$$

We solve for largest gain parameter, ω_0, that ensures that the roots of DT are inside the unit circle. Robust stability obtains if $\omega_0 > 0.01$, i.e. we have stability under constant gain learning for all $\omega < 0.01$. The numerical results are given in Table II in the main text. For those calculations, we use the calibration reported in section 3 with $\gamma = 128.21$. For simplicity, the shock is assumed to be i.i.d.[52]

[52]Mathematica routine available on request.

B.3 Flexible price case

Consider the linearized IH model studied above:

$$\begin{pmatrix} R_t \\ \pi_t \end{pmatrix} = \sum_{j=1}^{\infty} \beta^j \begin{pmatrix} -1 & \beta^{-2} \\ -\frac{\pi^*}{\psi_p} & \beta^{-2}\frac{\pi^*}{\psi_p} \end{pmatrix} \begin{pmatrix} R^e_{t+j} \\ \pi^e_{t+j} \end{pmatrix} + \sum_{i=1}^{L-1} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} R_{t-i} \\ \pi_{t-i} \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ \frac{\pi^*}{\psi_p}\epsilon_t \end{pmatrix}. \tag{82}$$

This model can be put in the form (77) where the 2×2 matrices of the model take the form

$$M = (m_{ij}),\ \hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \mathcal{A}_i \equiv \mathcal{M}_i + \hat{N} = \begin{pmatrix} 0 & a_{i12} \\ 0 & a_{i22} \end{pmatrix} \text{ for } i = 1, \ldots, N-1, \tag{83}$$

where $m_{11} = -1$, $m_{12} = \beta^{-2}$, $m_{21} = -\pi^*/\psi_p$, $m_{22} = \beta^{-2}\pi^*/\psi_p$. Postulating the PLM with matrices A_i, $i = 1, N-1$, it is possible in principle to compute the unprojected and projected ALMs. For example, we can show that both ALMs yield non-stationary processes for any set of PLM parameters when the model is calibrated with $\beta = 0.99$, $\psi_p = 1.2$, $\pi^* = 1.005$, $L = 4$ and $N = L - 1$. With these parameter values and augmenting the ALM with $\hat{N}$ to $VAR(3)$ process it is seen that the projected ALM process is non-stationary.

Proposition 4 in Section 3.4 establishes non-stationarity under more general assumptions about the flexible price model. The proposition covers the general case $N < L - 1$. The proof of the proposition is in Appendix D.

C Domain of Escape for Inflation Targeting and AIT

Figure A.1 shows the domain of escape under IT. The basic parameter settings are as given earlier.

Figure A.1 HERE

It should be noted that the result about escape from low steady state π_{Low}, y_{Low} differs from that in Figure I of Honkapohja and Mitra (2020). There are some differences in parameter values and most importantly in initial conditions for R_0 and R_0^e. In computing conditional domain of attraction it is natural to assume that R_0 and R_0^e are approximately equal to the steady state value R^*, whereas computation of domain of escape Figure A.1 assumes that R_0 and R_0^e are approximately 1.

Figure A.2 shows the domain of escape under AIT when $\tilde{A}_0 = \bar{A}$ where $\bar{A}$ is the rational expectations equilibrium coefficients corresponding to the unique dynamically stable MSV solution of the linearized model (i.e. the model linearized around the target steady state).

Figure A.2 HERE

C.1 Robust Stability: Monetary Policy Parameters and Transparency

Tables C.1 and C.2 show how Table I results (full opacity) change if either $\psi_y = 0.125$ or $\psi_p = 2$. Table C.3 reports robust stability results for the case of learning with transparency. The approach outlined in Appendix B.2 with $N = L$ and $\tilde{A}_n$ containing the correct MSV coefficients was taken to produce the results reported in Table C.3. All other parameters are reported in section 3.

Table C.1: Least upper bounds for $\psi_y = 0.125$

γ	42	128.21	350
ω_0 (IT)	0.03434	0.08630	0.10540
ω_0 (PLT)	0.00743	0.00478	0.00274
ω_0 (AIT with $L = 6$)	0.00299	0.00437	0.00528
ω_0 (AIT with $L = 20$)	0.00024	0.00053	0.00097
ω_0 (AIT with $L = 32$)	0.00009	0.00021	0.00044

Table C.2: Least upper bounds for $\psi_p = 2$

γ	42	128.21	350
ω_0 (IT)	0.03569	0.04656	0.04816
ω_0 (PLT)	0.01283	0.00928	0.00650
ω_0 (AIT with $L = 6$)	0.00355	0.00517	0.00691
ω_0 (AIT with $L = 20$)	0.00032	0.00065	0.00117
ω_0 (AIT with $L = 32$)	0.00012	0.00027	0.00053

Table C.3: Least upper bounds under learning with transparency.

γ	42	128.21	350
ω_0 ($L = 6$)	0.02351	0.02797	0.03586
ω_0 ($L = 20$)	0.02343	0.02783	0.03508
ω_0 ($L = 32$)	0.02343	0.02783	0.03508

D Proofs

D.1 Proof of Proposition 1

In the linearization (30)-(31) we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\psi_p/\pi^* & 0 \end{pmatrix}, \ i = 1, ..., L-1.$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y*^{\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$\begin{aligned} M &= -(DF_x)^{-1}DF_{x^e} = \\ &\begin{pmatrix} \frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{\Game} & \frac{\pi^*y^*(\bar{g}-y^*)}{\Game} & \frac{\beta^2\pi^*y^*(y^*-\bar{g})}{\Game} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\Game} & \frac{\kappa\pi^*y^*(\bar{g}-y^*)}{\Game} & \frac{\beta^2\kappa\pi^*y^*(y^*-\bar{g})}{\Game} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*)}{\Game} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{\Game} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{\Game} \end{pmatrix}, \\ N_i &= -(DF_x)^{-1}DF_{x_i} = \\ &\begin{pmatrix} 0 & \frac{\beta\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\ 0 & \frac{\beta\kappa\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\ 0 & \frac{\pi^*\sigma\psi_p y^*}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \end{pmatrix}, \ i = 1, ..., L-1. \end{aligned}$$

where

$$\Game = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0.$$

Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system (30), (31)

$$Z_t = QZ_{t-1}, \text{ where} \tag{84}$$

$$Z_t = \begin{pmatrix} x_t^e & x_t & x_{t-1} & x_{t-2} & \cdots & x_{t-(L-2)} \end{pmatrix}^T$$

$$Q = \begin{pmatrix} (1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\ (1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\ 0 & I_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_3 & 0 \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(1 + \omega - \lambda)\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1 - \lambda)^2(\lambda^{L-1} + h \sum_{k=0}^{L-2} \lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*} \in (0, 1)$$

if $\gamma > 0$. Using the stability criterion in Jury (1961), the roots of $(\lambda^{L-1} + h \sum_{k=0}^{L-2} \lambda^k)$ are inside the unit circle if and only if[53]

$$1 - \frac{kh^2}{1 + (k-1)h} > 0, k = 1, \ldots, L,$$

which is satisfied for all L. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda, \omega) &= \tilde{P}(\lambda_0, \omega_0) + \tilde{P}_\lambda(\lambda_0, \omega_0)d\lambda + \tilde{P}_\omega(\lambda_0, \omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned}\tilde{P}_\omega(\lambda_0, \omega_0) &= 0 \\ \tilde{P}_\lambda(\lambda_0, \omega_0) &= 0\end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda, \omega) = \tilde{P}(\lambda_0, \omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda, \omega) - \tilde{P}(\lambda_0, \omega_0) = 0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0, \omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)} + \frac{\tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

[53]Proof in Mathematica available on request.

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= (-1)^L \frac{2(y^* - \bar{g})((1-\beta)\beta\pi^*\psi_y + \kappa y^*(L\beta\psi_p - \pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L \frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + L\beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*} \\
\tilde{P}_{\lambda\omega}(1,0) &= (-1)^L \frac{\kappa y^*(y^*-\bar{g})(\pi^* - 2L\beta\psi_p) + (2-\beta)\beta\pi^*\psi_y(\bar{g}-y^*) - (1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}
\end{aligned}$$

One can show that $P_{\omega\omega}(1,0) > 0$, $P_{\lambda\lambda}(1,0) > 0$, $P_{\lambda\omega}(1,0) > 0$ if L is even and $P_{\omega\omega}(1,0) < 0$, $P_{\lambda\lambda}(1,0) < 0$, $P_{\lambda\omega}(1,0) < 0$ if L is odd. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for small ω and $\kappa > 0$.

D.2 Proof of Proposition 2

In the case $\gamma = 0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. Introducing the notation $\tilde{x}_t = (\pi_t, R_t)^T$, the linearization (30)-(31) becomes

$$\begin{aligned}
\tilde{M} &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*}{\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{\psi_p(1-\beta)} \\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)} \end{pmatrix} \text{ and} \\
\tilde{N}_i &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i}. = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \; i = 1, ..., L-1.
\end{aligned}$$

The system becomes

$$\tilde{Z}_t = \tilde{Q}\tilde{Z}_{t-1}, \text{ where} \tag{85}$$

$$\tilde{Z}_t = \begin{pmatrix} \tilde{x}_t^e & \tilde{x}_t & \tilde{x}_{t-1} & \tilde{x}_{t-2} & \cdots & \tilde{x}_{t-(L-2)} \end{pmatrix}^T$$

$$\tilde{Q} = \begin{pmatrix} (1-\omega)I_2 & \omega I_2 & 0 & \cdots & 0 & 0 \\ (1-\omega)\tilde{M} & \omega\tilde{M} + \tilde{N}_1 & \tilde{N}_2 & \cdots & \tilde{N}_{L-2} & \tilde{N}_{L-1} \\ 0 & I_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_2 & 0 \end{pmatrix}.$$

Note that in $\tilde{Q}$ we have $\tilde{N}_i = \tilde{N}$ for all i and $\tilde{N}$ is zero except for element $\tilde{n}_{11}$. In the determinant eliminate the second column from each block ≥ 3 and also corresponding row. We get

$$\det[\tilde{Q} - \lambda I_{2L}] = (-\lambda)^{L-2}\det[\tilde{K}_{L+2}], \tag{86}$$

where

$$\tilde{K}_{L+2} = \begin{bmatrix} (1-\omega)I_2 - \lambda I_2 & \omega I_2 & 0 & 0 & \cdots & 0 & 0 \\ (1-\omega)\tilde{M} & \omega\tilde{M} + \tilde{N}_1 - \lambda I_2 & N1 & N1 & \cdots & N1 & N1 \\ 0 & (1,0) & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}_{(L+2)x(L+2)}$$

and $N1 = (-1, 0)^T$.

Consider first the case $L = 1$, so there are no lags. We can focus on the learning dynamics of π_t and R_t, i.e. the matrix

$$\begin{pmatrix} (1-\omega)I & \omega I \\ (1-\omega)\tilde{M} & \omega\tilde{M} \end{pmatrix}, \text{ where } \tilde{M} = \begin{pmatrix} \frac{-\pi^*}{(\beta-1)\beta\psi_p} & \frac{\beta\pi^*}{(\beta-1)\psi_p} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}.$$

Assume $\psi_p > \beta^{-1}\pi^* = \bar{R}$. When $L = 1$ the system is four dimensional and two of the eigenvalues are those of $\tilde{M}$. Clearly $tr(\tilde{M} - I) < 0$ and $\det(\tilde{M} - I) > 0$. The other two eigenvalues are a repeated root equal to $1 - \omega < 1$ for all small ω. So E-stability holds in this case.

In the case of general L, the characteristic polynomial of K_{L+2} has the following structure:[54]

$$\det[K_{L+2}] = \lambda(\lambda - 1 + \omega)P(n, \omega, \lambda)$$

where $n = L$ and

$$\begin{aligned} P(n,\omega,z) &= z^n + \tilde{b}z^{n-1} + \tilde{c}z^{n-2} + ... + \tilde{c}z + a_n, \text{ where} \qquad (87) \\ \tilde{b} &= \frac{\omega}{1-\beta}b_1 \text{ with } b_1 = (1 - \frac{\pi^*}{\beta\psi_P}), \\ \tilde{c} &= \frac{\omega}{1-\beta} \text{ and } a_n = \tilde{c} - 1. \end{aligned}$$

Here $b_1, \beta, \omega \in (0, 1)$. To apply the Schur-Cohn conditions the polynomial (87) is written in a general form

$$A(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$$

so

$$a_1 = \tilde{b}, a_2 = \tilde{c}, \ldots, a_{n-1} = \tilde{c} \text{ and } a_n = \tilde{c} - 1. \qquad (88)$$

Then define the matrices[55]

$$D^{\pm}_{n-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_n & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_n & \cdots & a_4 & a_3 \\ a_n & a_{n-1} & \cdots & a_3 & a_2 \end{pmatrix}.$$

The roots of $A(\lambda)$ are inside the unit circle if and only if the following conditions hold: (a) $A(1) > 0$ and (b) $(-1)^n A(-1) > 0$, and (c) the matrices $D^{\pm}_{n-1}$ are positive innerwise, i.e., $|D^{\pm}_1| > 0$, $|D^{\pm}_3| > 0, ..., |D^{\pm}_{n-1}| > 0$ when n is even and $|D^{\pm}_2| > 0$, $|D^{\pm}_4| > 0, ..., |D^{\pm}_{n-1}| > 0$ when n is odd, where $D^{\pm}_i > 0$ denote the inners of $D^{\pm}_{n-1} > 0$ as defined in Elaydi (2005).

Now

$$\begin{aligned} A(1) &= 1 + \tilde{b} + (n-2)\tilde{c} + \tilde{c} - 1 = \frac{\omega}{1-\beta}(b_1 + n - 1) > 0 \text{ if } n \geq 1. \\ (-1)^n A(-1) &= 1 - \tilde{b} + \tilde{c} - 1 = \tilde{c} - \tilde{b} = \frac{\omega}{1-\beta}(1 - b_1) > 0 \text{ if } n \text{ is even and} \\ (-1)^n A(-1) &= (-1)(-1 + \tilde{b} - \tilde{c} + \tilde{c} - 1) = 2 - \tilde{b} > 0 \text{ if } n \text{ is odd,} \end{aligned}$$

[54]The Mathematica routine is available on request.

[55]This form of the Schur-Cohn conditions is given in Elaydi (2005), p.247.

so conditions (a) and (b) hold. Substituting in the relations (88) we get

$$D_{n-1}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \tilde{b} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \tilde{c} & \tilde{c} & \cdots & 1 & 0 \\ \tilde{c} & \tilde{c} & \cdots & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & \tilde{c}-1 \\ 0 & 0 & \cdots & \tilde{c}-1 & \tilde{c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \tilde{c}-1 & \cdots & \tilde{c} & \tilde{c} \\ \tilde{c}-1 & \tilde{c} & \cdots & \tilde{c} & \tilde{c} \end{pmatrix},$$

where $n \geq 2$. Assume first that n is even and consider the inners $D_1^{\pm}$, $D_3^{\pm}$, ..., $D_{n-1}^{\pm}$. we get

$$\left|D_1^{\pm}\right| = 1 \pm (\tilde{c}-1) = \tilde{c} \text{ or } 2-\tilde{c} \text{ which are } > 0 \text{ if } \tilde{c} < 2$$

and

$$\left|D_3^{\pm}\right| = \left| \begin{pmatrix} 1 & 0 & 0 \\ \tilde{b} & 1 & 0 \\ \tilde{c} & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & a_n \\ 0 & a_n & \tilde{c} \\ a_n & \tilde{c} & \tilde{c} \end{pmatrix} \right|$$

yielding $\left|D_3^{-}\right| = (\tilde{b}-\tilde{c})(\tilde{c}+\tilde{b}-\tilde{b}\tilde{c}) < 0$ as $0 < \tilde{b} < \tilde{c}$, which implies instability for $n \geq 4$ even, but stability for $n = 2$.

Next assume that n is odd. One computes

$$\left|D_2^{\pm}\right| = \left| \begin{pmatrix} 1 & \pm a_n \\ \tilde{b} \pm a_n & 1 \pm \tilde{c} \end{pmatrix} \right|$$

so $\left|D_2^{+}\right| = 3\tilde{c} - b\tilde{c} + \tilde{b} - \tilde{c}^2 > 0$ and $\left|D_2^{-}\right| = (\tilde{c}-\tilde{b})(1-\tilde{c}) > 0$ for sufficiently small $\tilde{c} > 0$, which implies stability for $n = 3$. Next consider

$$D_4^{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{b} & 1 & 0 & 0 \\ \tilde{c} & \tilde{b} & 1 & 0 \\ \tilde{c} & \tilde{c} & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 0 & \tilde{c}-1 \\ 0 & 0 & \tilde{c}-1 & \tilde{c} \\ 0 & \tilde{c}-1 & \tilde{c} & \tilde{c} \\ \tilde{c}-1 & \tilde{c} & \tilde{c} & \tilde{c} \end{pmatrix}.$$

One computes

$$D_4^{-} = \begin{pmatrix} 1 & 0 & 0 & 1-\tilde{c} \\ \tilde{b} & 1 & 1-\tilde{c} & -\tilde{c} \\ \tilde{c} & 1+\tilde{b}-\tilde{c} & 1-\tilde{c} & -\tilde{c} \\ 1 & 0 & \tilde{b}-\tilde{c} & 1-\tilde{c} \end{pmatrix},$$

so $\left|D_4^{-}\right| = (\tilde{b}-\tilde{c})^2(\tilde{b}\tilde{c}-\tilde{b}-1) < 0$ implying instability for n odd.

So overall there is instability for $n \geq 4$ and stability for $n \leq 3$.

Proof of the Remark in Section 1.1: Recalling (8), its characteristic polynomial is

$$P(n,\omega,z) = z^n + (1-M)\omega z^{n-1} + \omega z^{n-2} + ... + \omega z + (\omega - 1),$$

where $n = L$. This polynomial has same form as (87) except that the coefficients are now $\tilde{b} = \omega(1-M)$ and $\tilde{c} = \omega$. The rest of the argument follows the preceding proof.

D.3 Proof of Proposition 3

First consider (i). The model (18)-(20) with i.i.d. government spending shock can be written as

$$\begin{aligned} y_t &= c_R R_t + \sum_{j=1}^{\infty} \beta^j \left(c_\pi \pi_{t+j}^e + c_y y_{t+j}^e + c_R R_{t+j}^e \right) + \hat{\tilde{g}}_t \\ \pi_t &= \kappa y_t + \kappa \sum_{j=1}^{\infty} \beta^j y_{t+j}^e + \frac{(1-\nu) y^* \sigma}{\gamma (y^* - \bar{g})^{\sigma+1}} \hat{\tilde{g}}_t \\ R_t &= \frac{\psi_p}{\pi^*} \sum_{i=0}^{L-1} \pi_{t-i} + \psi_y \frac{y_t}{y^*} \end{aligned}$$

where $\hat{\tilde{g}}_t$ is an i.i.d shock, $c_R = -c^* \beta (\sigma \pi^*)^{-1}$, $c_\pi = -c_R \beta^{-2}$, $c_y = \beta^{-1}(1-\beta)$, and $\kappa = \kappa(\gamma)$ with $\frac{\partial \kappa}{\partial \gamma} < 0$ and $\lim_{\gamma \to \infty} \kappa = 0$. Suppose the following PLM:

$$\pi_t = a_\pi + b_\pi \pi_{t-1} \tag{89}$$
$$R_t = a_R + b_R \pi_{t-1} \tag{90}$$
$$y_t = a_y + b_y \pi_{t-1} \tag{91}$$

which implies

$$\begin{aligned} \pi_{t+j}^e &= (1 - b_\pi^{j+1}) a_\pi / (1 - b_\pi) + b_\pi^{j+1} \pi_{t-1} \\ s_{t+j}^e &= a_s + b_s (1 - b_\pi^j) a_\pi / (1 - b_\pi) + b_s b_\pi^j \pi_{t-1} \end{aligned}$$

where $s = y, R$. Under this PLM, a restricted perceptions equilibrium (RPE) of (18)-(20) is given by coefficients $(a_\pi, a_R, a_y, b_\pi, b_R, b_y)$ which satisfy the least-squares orthogonality restriction

$$E \pi_{t-1} \left(s_t - a_s - b_s \pi_{t-1} \right) = E (\pi_{t-1} - a_s) \left(s_t - a_s - b_s \pi_{t-1} \right) = 0$$

for $s = \pi, y, R$, where $|b_\pi| < 1$, $b_\pi \neq 0$, and E denotes the unconditional expectations operator. This restriction implies: $a_s = 0$.[56]

Suppose an RPE exists and that agents update their estimates of (a_y, a_R, a_π) according to (34) with b_π, b_R, b_y fixed to their RPE values. Substituting expectations into the system (18)-(20) gives the first-order system:

$$Z_t = Q Z_{t-1} + \tilde{\epsilon}_t. \tag{92}$$

where $Z_t = (\pi_t, R_t, y_t, a_{\pi,t}, a_{R,t}, a_{y,t}, \pi_{t-1}, \ldots, \pi_{t-L+2})$. For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{L+4}]$ must be inside the unit circle. One can show that for sufficiently large γ (i.e. in the limit $\kappa \to 0$), $P(\lambda) = (1 - \frac{c^* \omega \beta \psi_y}{(1-\beta)(y^* \pi^* \sigma + c^* \beta \psi_y)} - \lambda) \lambda^{L+1} (\lambda - (1 - \omega))$. Hence, we have stability for small κ and small ω if $\psi_y > 0$. To show stability for small κ and small ω when $\psi_y = 0$, we compute the total differential of $P(\lambda)$:

$$\begin{aligned} \frac{\partial P}{\partial \lambda} d\lambda + \frac{\partial P}{\partial \kappa} d\kappa &= 0 \\ \implies \quad \frac{d\lambda}{d\kappa} &= -\frac{\partial P}{\partial \kappa} / \frac{\partial P}{\partial \lambda}. \end{aligned}$$

[56] Mathematica routine available on request.

Evaluated at $\lambda = 1$ and $\kappa = \psi_y = 0$ we have

$$\implies \frac{d\lambda}{d\kappa} = \frac{c^*\omega(\pi^* - L\beta\psi_p)}{(\beta-1)^2(\pi^*)^2\sigma} < 0$$

Therefore, an RPE is stable under constant gain learning about the intercept term, assuming an RPE exists and κ sufficiently small.

Now consider (ii). In the limit $\gamma \to 0$, $y_t = g_y \hat{\tilde{g}}_t$ where g_y is a complicated function of deep structural parameters (see Appendix A.6) and therefore (18)-(20) reduces to:[57]

$$R_t = \sum_{j=1}^{\infty} \beta^j \left(\beta^{-2}\pi^e_{t+j} - R^e_{t+j}\right) + \epsilon_t, \quad (93)$$

$$R_t = \frac{\psi_p}{\pi^*} \sum_{i=0}^{L-1} \pi_{t-i}, \quad (94)$$

where ϵ_t is proportional to the i.i.d government spending shock $\tilde{g}_t$, and $\psi_y = 0$ is assumed for simplicity. Suppose the PLM:

$$\pi_t = a_\pi + b_\pi \pi_{t-1}, \quad (95)$$

$$R_t = a_R + b_R \pi_{t-1}, \quad (96)$$

which implies

$$\pi^e_{t+j} = (1 - b_\pi^{j+1}) a_\pi / (1 - b_\pi) + b_\pi^{j+1} \pi_{t-1},$$

$$R^e_{t+j} = a_R + b_R(1 - b_\pi^j) a_\pi / (1 - b_\pi) + b_R b_\pi^j \pi_{t-1}.$$

Under the PLM, (95)-(96), a restricted perceptions equilibrium (RPE) of (93)-(94) is given by coefficients (a_π, a_R, b_π, b_R) which satisfy the least-squares orthogonality restriction

$$E\pi_{t-1}(\pi_t - a_\pi - b_\pi\pi_{t-1}) = E(\pi_{t-1} - a_\pi)(\pi_t - a_\pi - b_\pi\pi_{t-1}) = 0,$$

$$E\pi_{t-1}(R_t - a_R - b_R\pi_{t-1}) = E(\pi_{t-1} - a_R)(R_t - a_R - b_R\pi_{t-1}) = 0,$$

where $|b_\pi| < 1$, $b_\pi \neq 0$, and E denotes the unconditional expectations operator. This restriction implies: $a_R = a_\pi = 0$, $b_R = \frac{b_\pi^2}{\beta}$.[58]

Suppose an RPE exists and that agents update their estimates of (a_R, a_π) according to (34) with b_π and b_R fixed to their RPE values. Substituting expectations (34) and (94) into (93) yields:

$$\begin{aligned}
\pi_t &= d_\pi a_{\pi,t} + d_R a_{R,t} + e_\pi \pi_{t-1} - \sum_{i=2}^{L-1} \pi_{t-i} + \tilde{\epsilon}_t \\
&= (e_\pi + d_\pi\omega)\pi_{t-1} + (-d_\pi b_\pi \omega - d_R b_R \omega - 1)\pi_{t-2} \\
&+ d_R \omega R_{t-1} + d_\pi(1-\omega) a_{\pi,t-1} + d_R(1-\omega) a_{R,t-1} - \sum_{i=3}^{L-1} \pi_{t-i} + \tilde{\epsilon}_t, \quad (97)
\end{aligned}$$

[57]Here, as with other flexible price results, we assume agents learn the exogenous process for output, i.e. $y_t = g_y \hat{\tilde{g}}_t$ which implies $y^e_{t+j} = 0$ for all $j \geq 1$.

[58]Mathematica routine available on request. Given $a_R = a_\pi = 0$, we can show that $R_t = B(b_R)\pi_{t-1} + r\epsilon_t$ where r is a scalar, $B(b_R) = \frac{\psi_p}{\pi^*}(e_\pi + 1)$ and e_π is defined in (97). Therefore, $\frac{E(R_t\pi_{t-1})}{E(\pi_t\pi_t)} = B(b_R) = b_R$ since $a_R = 0$. Solving $B(b_R) = b_R$ for b_R gives $b_R = \frac{b_\pi^2}{\beta}$.

where

$$
\begin{aligned}
e_\pi &= \frac{\pi^* b_\pi (b_\pi \beta^{-1} - b_R \beta)}{\psi_p (1-\beta b_\pi)} - 1, \\
d_\pi &= \frac{\pi^*}{\psi_p (1-b_\pi)} \left(\frac{\beta^{-1} - \beta b_R}{1-\beta} - \frac{b_\pi \left(\beta^{-1} b_\pi - \beta b_R\right)}{1-\beta b_\pi} \right), \\
d_R &= -\frac{\pi^* \beta}{\psi_p (1-\beta)}.
\end{aligned}
$$

Introduce the notation $Z_t = (\pi_t, R_t, a_{\pi,t}, a_{R,t}, \pi_{t-1}, \ldots, \pi_{t-L+2})$. Modifying the system gives

$$Z_t = Q Z_{t-1} + \hat{\epsilon}_t. \tag{98}$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{L+2}]$ must be inside the unit circle. One can show that in the limit $\omega \to 0$

$$P(\lambda) = -(1-\lambda)^2 \lambda \tilde{P}(\lambda)$$

Thus, some roots of $P(\lambda)$ are outside of the unit circle if any root of $\tilde{P}(\lambda)$ is outside the unit circle where

$$\tilde{P}(\lambda) = \lambda^{L-1} + \left(1 - \frac{\pi^* b_\pi^2}{\beta \psi_p}\right) \lambda^{L-2} + \sum_{k=0}^{L-3} \lambda^k \tag{99}$$

where $0 < \frac{\pi^* b_\pi^2}{\beta \psi_p} < 1$ under the Taylor Principle with $\pi^* \geq 1$. The preceding equation has the following form

$$Q(\lambda) = \lambda^n + (1-c)\lambda^n + \sum_{k=0}^{n-2} \lambda^k$$

where $c \in (0,1)$ and $n = L-1$. We can assess stability following the Schur-Cohn conditions presented in section C.2.

If n is even. Then

$$
\begin{aligned}
\left|D_3^{\pm}\right| &= \left| \begin{pmatrix} 1 & 0 & 0 \\ 1-c & 1 & 0 \\ 1 & 1-c & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right| \\
& \quad -2c + c^2 \text{ or } -c^2
\end{aligned}
$$

so $\left|D_3^{+}\right| < 0$ and $\left|D_3^{-}\right| < 0$, which implies instability.

Then consider the case n is odd.

$$\left|D_2^{\pm}\right| = \left| \begin{pmatrix} 1 & \pm 1 \\ 1-c\pm 1 & 1 \pm 1 \end{pmatrix} \right| = (1 \pm 1) \mp (1 - c \pm 1) = c \text{ or } -c.$$

So there is instability as $\left|D_2^{-}\right| < 0$. So overall there is instability for $L \geq 4$.

D.4 Proof of Proposition 4

It is seen that only lags of inflation appear in structural model (82). Its coefficient matrices $\mathcal{A}_i$ take the form given in (83), and the resulting temporary equilibrium is $VAR(L-1)$. Let $y_t = (R_t, \pi_t)^T$. The temporary equilibrium system (unprojected ALM) takes the form

$$y_t = \mathcal{A}_1 y_{t-1} + ... + \mathcal{A}_{N-1} y_{t-(N-1)} + \hat{N} y_{t-N} + ... + \hat{N} y_{t-(L-1)} + z_t. \tag{100}$$

The relevant characteristic polynomial is $\tilde{H}(\lambda) = \lambda^{L-1} H(\lambda)$, where

$$H(\lambda) = \lambda^{L-1} - \sum_{i=1}^{N-1} a_{i22} \lambda^{L-1-i} + \sum_{i=N}^{L-1} \lambda^{L-1-i},$$

The temporary equilibrium system is stationary if the roots of $H(\lambda)$ are inside the unit circle. The Schur-Cohn conditions are the relevant stability conditions. According to Proposition 5.1 in Elaydi (2005), condition (iii) is necessary for $H(\lambda)$ to have all of its roots inside the unit circle. This condition is stated in terms of the inners of the following matrices

$$B_{L-2}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_{122} & 1 & 0 & \cdots & 0 \\ -a_{222} & -a_{122} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{L-3} & b_{L-4} & \cdots & -a_{122} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & b_{L-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & b_4 & b_3 \\ 1 & b_{L-2} & \cdots & b_3 & -a_{222} \end{pmatrix}.$$

where $b_k = -a_{k22}$ if $N > k$ or where $b = 1$ if $N \leq k$. The smallest inner of $B_{L-2}^{\pm}$ is either

$$\left| B_1^{\pm} \right| = 1 \pm 1 = 2 \text{ or } 0$$

if L is odd or

$$\begin{aligned} \left| B_2^{\pm} \right| &= \left| \begin{pmatrix} 1 & 0 \\ -a_{122} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ 1 & b_{L-2} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 1 & 1 \\ 1 - a_{122} & 1 + b_{L-2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 \\ -a_{122} - 1 & 1 - b_{L-2} \end{pmatrix} \right| \\ &= b_{L-2} + a_{122} \text{ or } -(b_{L-2} + a_{122}). \end{aligned}$$

where $b = -a_{(L-2)22}$ or -1 if L is even. So there is always a zero inner, which implies that not all roots are inside the unit circle.

D.5 Proof of Proposition 5

The dynamic model is given by a linearized system of the form (30) and (31) where the interest rate is now (37). In this proof, we set $\psi_p = \psi_p / (\sum_{i=0}^{L-1} \mu^i)$ (i.e. we write the averaging constant explicitly in the interest rate rule, but this detail is not essential for the results). Again in the limit $\gamma \to 0$ the first equation is independent from the rest of the system and output expectations y_t^e are convergent. Separating the equation for y_t^e, the state variables are $\tilde{x}_t = (\pi_t, R_t)^T$ and the linearized system is of the form (85) but the coefficient matrices $\tilde{M}$ and $\tilde{N}_i$ change to

$$\begin{aligned}
\tilde{M} &= -(D\tilde{F}_x)^{-1} D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)}{\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)}{\psi_p(1-\beta)} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}, \\
\tilde{N}_i &= -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i} = \begin{pmatrix} -\mu^i & 0 \\ 0 & 0 \end{pmatrix}, \; i = 1, ..., L-1.
\end{aligned}$$

and the system is now

$$\tilde{Z}_t = \tilde{Q}_2 \tilde{Z}_{t-1}, \tag{101}$$

where $\tilde{Z}_t$ is defined in the proof of Proposition 2, but $\tilde{Q}_2$ incorporates the new forms of $\tilde{M}$ and $\tilde{N}_i$ in $\tilde{Q}$. Consider the characteristic polynomial of $\tilde{Q}_2$ of (101)

$$\det[\tilde{Q}_2 - \lambda I_{2L}] = 0. \tag{102}$$

Given that the second columns of $\tilde{N}_i$ are zero vectors, the determinant in (102) has $L-2$ roots equal to zero. Then analyzing the remaining $(L+2)$ dimensional determinant, again it turns out that there is one more zero root and one root equal to $1-\omega$. Factoring out these, we are left with a polynomial of degree L. Introducing more familiar notation $n = L$, the polynomial is

$$P_2(n, \omega, \lambda) = \lambda^n + b(\omega)\lambda^{n-1} + a(\omega)[\mu\lambda^{n-2} ... + \mu^{n-2}\lambda] + (a(\omega)\mu^{n-1} - \mu^n), \tag{103}$$

where μ is the weight parameter in (37),

$$a(\omega) = \frac{\omega}{1-\beta} + \mu - 1, b(\omega) = a(\omega) - \frac{\omega}{1-\beta} b_1 \text{ with } b_1 = \frac{\pi^*\left(\sum_{i=0}^{n-1}\mu^i\right)}{\beta\psi_p}$$

and where $\pi^* < \beta\psi_P$ and $n \geq 2$ are assumed. We again consider how any root varies as ω varies from 0 to small values $d\omega > 0$ and require that in this variation the root is continuously a root of the characteristic polynomial. If $\omega \to 0$ we have $a(\omega) \to \mu - 1$ and $b(\omega) \to \mu - 1$, so the characteristic equation becomes

$$(1-\lambda)(\lambda^{n-1} + \mu\lambda^{n-2} + \mu^2\lambda^{n-3} + ... + \mu^{n-2}\lambda + \mu^{n-1}). \tag{104}$$

There is one root of unity. For the other roots one can apply a generalization of the classic Enerstrom-Kakeya theorem in Gardner and Govil (2014), Theorem 3.6, stating that the other roots of the polynomial in (104) satisfy $|\lambda| < \mu < 1$.

Then consider the root of 1. Assume now a small perturbation $\omega > 0$. By continuity of eigenvalues the $n-1$ roots that are approximate to the roots of the latter polynomial in (104) remain inside the unit circle. To determine whether the unit root contributes to stability we compute the partial derivatives

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= n\lambda^{n-1} + (n-1)b(\omega)\lambda^{n-2} + a(\omega)[\mu(n-2)\lambda^{n-3} ... + \mu^{n-2}], \\
\frac{\partial P_2}{\partial \omega} &= +b'(\omega)\lambda^{n-1} + a'(\omega)[\mu\lambda^{n-2} + ... + \mu^{n-2}\lambda] + a'(\omega)\mu^{n-1}.
\end{aligned}$$

At $\omega = 0$ and $\lambda = 1$ we have

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= \frac{1-\mu^n}{1-\mu} > 0, \\
\frac{\partial P_2}{\partial \omega} &= \frac{1}{1-\beta}\left(1 - \frac{\pi^* \sum_{k=0}^{n-1} \mu^k}{\beta \psi_p} + \mu \frac{1-\mu^{n-1}}{1-\mu}\right) > 0,
\end{aligned}$$

since $a'(0) = (1-\beta)^{-1}$ and $b'(0) = (1-\beta)^{-1}(1-b_1)$. Then taking the differential of (103) and requiring

$$\frac{\partial P_2}{\partial \omega} d\omega + \frac{\partial P_2}{\partial \lambda} d\lambda = 0 \implies \frac{\partial \lambda}{\partial \omega} < 0.$$

So for small $\omega > 0$ the real root corresponding to limit 1 is inside the unit circle.

Next consider part (ii) of the proposition. In the linearization we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_{-i}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\mu^i\psi_p}{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)} & 0 \end{pmatrix}, \; i = 1, ..., L-1,$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y*^{\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0.$$

It follows that

$$\begin{aligned}
M &= -(DF_x)^{-1}DF_{x^e} = \\
&\begin{pmatrix} \frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)+(\beta-1)(\pi^*)^2\sigma\right)}{\eth_2} & \frac{\pi^*y^*(\bar{g}-y^*)}{\eth_2} & \frac{\beta^2\pi^*y^*(y^*-\bar{g})}{\eth_2} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\eth_2} & \frac{\kappa\pi^*y^*(\bar{g}-y^*)}{\eth_2} & \frac{\beta^2\kappa\pi^*y^*(y^*-\bar{g})}{\eth_2} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*/\left(\sum_{i=0}^{L-1}\mu^i\right))}{\eth_2} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{\eth_2} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{\eth_2} \end{pmatrix}, \\
N_i &= -(DF_x)^{-1}DF_{x_i} = \\
&\begin{pmatrix} 0 & \frac{\mu^i\beta\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\beta\kappa\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\pi^*\sigma\psi_p y^*(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \end{pmatrix}, \; i = 1, ..., L-1.
\end{aligned}$$

where

$$\eth_2 = (\beta-1)\left(\pi^*(\beta\psi_y(y^*-\bar{g}) + \pi^*\sigma y^*) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)\right) < 0.$$

The system is now like (84)

$$Z_t = Q_2 Z_{t-1}, \tag{105}$$

where Z_t is as before in Proposition 1, but Q_2 incorporates the new forms of M and N_i. Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system yields

$$Z_t = \begin{pmatrix} x_t^e & x_t & x_{t-1} & x_{t-2} & \cdots & x_{t-(L-2)} \end{pmatrix}^T$$

$$Q_2 = \begin{pmatrix} (1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\ (1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\ 0 & I_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_3 & 0 \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q_2 - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(1+\omega-\lambda)\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1-\lambda)^2(\lambda^{L-1} + h\mu \sum_{k=0}^{L-2} \mu^{L-2-k}\lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta\kappa\psi_p y^*(y^* - \bar{g}) + (\beta\pi^*\psi_y(y^* - \bar{g}) + (\pi^*)^2\sigma y^*)\left(\sum_{i=0}^{L-1} \mu^i\right)} \in (0,1)$$

The polynomial has two unit roots. For the other roots one can apply a generalization of the classic Enerstrom-Kakeya theorem in Gardner and Govil (2014), Theorem 3.6, stating that the roots of the second polynomial in $\tilde{P}(\lambda)$ satisfy $|\lambda| < \mu < 1$.

Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned} \tilde{P}(\lambda,\omega) &= \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_\lambda(\lambda_0,\omega_0)d\lambda + \tilde{P}_\omega(\lambda_0,\omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + Q, \end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned} \tilde{P}_\omega(\lambda_0,\omega_0) &= 0 \\ \tilde{P}_\lambda(\lambda_0,\omega_0) &= 0 \end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda,\omega) = \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda,\omega) - \tilde{P}(\lambda_0,\omega_0) = 0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)} + \frac{\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Evaluating the partial derivatives at $(\lambda_0,\omega_0) = (1,0)$ we have

$$\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= (-1)^L \frac{2(y^*-\bar{g})((1-\beta)\beta\pi^*\psi_y + \kappa y^*(\beta\psi_p - \pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L \frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*} \\
\tilde{P}_{\lambda\omega}(1,0) &= (-1)^L \frac{\kappa y^*(y^*-\bar{g})(\pi^*-2\beta\psi_p) + (2-\beta)\beta\pi^*\psi_y(\bar{g}-y^*) - (1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*\right)}
\end{aligned}$$

One can show that $\tilde{P}_{\omega\omega}(1,0) > 0$, $\tilde{P}_{\lambda\lambda}(1,0) > 0$, $\tilde{P}_{\lambda\omega}(1,0) > 0$ if L is even and $\tilde{P}_{\omega\omega}(1,0) < 0$, $\tilde{P}_{\lambda\lambda}(1,0) < 0$, $\tilde{P}_{\lambda\omega}(1,0) < 0$ if L is odd. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for $\kappa \geq 0$ and small ω.

D.6 Proof of Proposition 6

In the linearization (41) we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{w_c\psi_p}{\pi^*} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta\kappa}{\beta-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{cb} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -(1-w_c)\psi_p/\pi^* & 0 \end{pmatrix}$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y^{*\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$
\begin{aligned}
M &= -(DF_x)^{-1} DF_{x^e} = \\
&\begin{pmatrix}
\frac{y^*\left(w_c\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{\eth_3} & \frac{\pi^* y^*(\bar{g}-y^*)}{\eth_3} & \frac{\beta^2\pi^* y^*(y^*-\bar{g})}{\eth_3} \\
\frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\eth_3} & \frac{\kappa\pi^* y^*(\bar{g}-y^*)}{\eth_3} & \frac{\beta^2\kappa\pi^* y^*(y^*-\bar{g})}{\eth_3} \\
\frac{\pi^*\sigma((\beta-1)\pi^*\psi_y - w_c\kappa\psi_p y^*)}{\eth_3} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+w_c\kappa\psi_p y^*)}{\eth_3} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+w_c\kappa\psi_p y^*)}{\eth_3}
\end{pmatrix}, \\
N &= -(DF_x)^{-1} DF_{cb} = \\
&\begin{pmatrix}
0 & \frac{(w_c-1)\beta\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0 \\
0 & \frac{(w_c-1)\beta\kappa\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0 \\
0 & \frac{(w_c-1)\pi^*\sigma\psi_p y^*}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0
\end{pmatrix} \\
N_{cb} &= \\
&\begin{pmatrix}
\frac{(w_c-1)\beta\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} \\
\frac{(w_c-1)\beta\kappa\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} \\
\frac{(w_c-1)\pi^*\sigma\psi_p y^*}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)}
\end{pmatrix}
\end{aligned}
$$

where $\eth_3 = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + w_c\beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0$.

Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system (31), (41), and the linearization of (39) yields yields

$$
Z_t = QZ_{t-1}, \text{ where} \tag{106}
$$

$$
Z_t = \begin{pmatrix} x_t & x_t^e & \pi_t^{cb} \end{pmatrix}^T
$$

$$
Q = \begin{pmatrix}
\omega M + w_c N & (1-\omega)M & (1-w_c)N_{cb} \\
\omega I_3 & (1-\omega)I_3 & 0_{3\times 1} \\
0 \;\; w_c \;\; 0 & \cdots & 1 - w_c
\end{pmatrix}.
$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_7]$ must be inside the unit circle. One can show that

$$
P(\lambda) = \lambda^3(1 + \omega - \lambda)\tilde{P}(\lambda).
$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$
\tilde{P}(\lambda) = (1 - \lambda)^2(\lambda - \mu)
$$

where $\mu = \frac{(1-w_c)\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y+\beta\kappa w_c\psi_p y^*)+(\pi^*)^2\sigma y^*} < \frac{\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y)+(\pi^*)^2\sigma y^*} = 1$. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$
\begin{aligned}
\tilde{P}(\lambda, \omega) &= \tilde{P}(\lambda_0, \omega_0) + \tilde{P}_\lambda(\lambda_0, \omega_0)d\lambda + \tilde{P}_\omega(\lambda_0, \omega_0)d\omega + \\
&\quad \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2} + Q,
\end{aligned}
$$

where subscripts denote partial derivatives and Q is a remainder.

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned} \tilde{P}_\omega(1,0) &= 0, \\ \tilde{P}_\lambda(1,0) &= 0, \end{aligned}$$

and imposing

$$\tilde{P}(\lambda, \omega) - \tilde{P}(\lambda_0, \omega_0) = 0,$$

we get the approximation

$$\tilde{P}_{\lambda\omega}(1,0) d\lambda dw + \tilde{P}_{\omega\omega}(1,0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(1,0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(1,0)}{\tilde{P}_{\lambda\omega}(1,0)} + \frac{\tilde{P}_{\lambda\lambda}(1,0)}{\tilde{P}_{\lambda\omega}(1,0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Further, we have

$$\begin{aligned} \tilde{P}_{\omega\omega}(1,0) &= \frac{2w_c(y^* - \bar{g})((\beta-1)\beta\pi^*\psi_y + \kappa\pi^* y^* - \beta\kappa\psi_p y^*)}{(1-\beta)^{-1}\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\ \tilde{P}_{\lambda\lambda}(1,0) &= -\frac{2w_c\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\ \tilde{P}_{\lambda\omega}(1,0) &= \frac{w_c y^*\left((\beta-1)(\pi^*)^2\sigma + (\beta-2)\beta\pi^*\psi_y + \kappa\pi^* y^* - 2\beta\kappa\psi_p y^*\right)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\ &- \frac{\bar{g}w_c((\beta-2)\beta\pi^*\psi_y + \kappa\pi^* y^* - 2\beta\kappa\psi_p y^*)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \end{aligned}$$

One can show that $\tilde{P}_{\omega\omega}(1,0) < 0$, $\tilde{P}_{\lambda\lambda}(1,0) < 0$, $\tilde{P}_{\lambda\omega}(1,0) < 0$ if $\beta\psi_p > \pi^*$. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for small w and $\kappa > 0$.

In part (ii) with $\gamma = 0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. The linearization (41) becomes

$$\begin{aligned} \tilde{M} &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*}{w_c\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{w_c\psi_p(1-\beta)} \\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)} \end{pmatrix} \text{ and} \\ \tilde{N} &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i}. = \begin{pmatrix} \frac{w_c-1}{w_c} & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\ \tilde{N}_{cb} &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i}. = \begin{pmatrix} \frac{w_c-1}{w_c} \\ 0 \end{pmatrix}, \; i = 1, ..., L-1. \end{aligned}$$

Introduce the notation $\tilde{x}_t = (\pi_t, R_t)$ etc. Modifying the system (31), (41) and the linearization of (39) yields

$$\tilde{Z}_t = \tilde{Q}\tilde{Z}_{t-1}, \text{ where} \tag{107}$$

$$\tilde{Z}_t = \begin{pmatrix} x_t & x_t^e & \pi_t^{cb} \end{pmatrix}^T$$

$$\tilde{Q} = \begin{pmatrix} \omega\tilde{M} + w_c\tilde{N} & (1-\omega)\tilde{M} & (1-w_c)\tilde{N}_{cb} \\ \omega I_2 & (1-\omega)I_2 & 0_{2\times1} \\ w_c \;\; 0 & \cdots & 1-w_c \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[\tilde{Q} - \lambda I_5]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^2(1 + \omega - \lambda)\tilde{P}(\lambda),$$

where

$$\tilde{P}(\lambda) = \lambda^2 + \frac{\beta w_c \psi_p(\beta + \omega - 1) - \pi^* \omega}{(1-\beta)\beta w_c \psi_p}\lambda + \frac{\pi^* \omega(1 - w_c)}{(1-\beta)\beta w_c \psi_p}.$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle.

Let $a_0 = \frac{\pi^* \omega(1-w_c)}{(1-\beta)\beta w_c \psi_p}$ and $a_1 = \frac{\beta w_c \psi_p(\beta+\omega-1)-\pi^* \omega}{(1-\beta)\beta w_c \psi_p}$. The roots of $\tilde{P}(\lambda)$ are inside the unit circle if and only if the Schur-Cohn condition, $|a_1| < 1 + a_0 < 2$, is satisfied. The Schur-Cohn condition is satisfied if $\psi_p > \max[\frac{\pi^*(\omega/w_c)(1-w_c)}{(1-\beta)\beta}, \bar{R}]$ and $\omega < \frac{(1-\beta)\beta w_c \psi_p}{\beta w_c \psi_p - \pi^*}$ if $\psi_p > \pi^*/(\beta w_c)$ or $\omega > 0$ otherwise.

References

Amano, R., S. Gnocchi, S. Leduc, and J. Wagner, "Average Is Good Enough: Average-Inflation Targeting and the ELB," Federal Reserve Bank of San Francisco Working Paper 2020-21 2020.

Andrade, B., J. Gali, H. Le Bihan, and J. Matheron, "Should the ECB Adjust its Strategy in the Face of a Lower r*?," *Journal of Economic Dynamics and Control*, 2021, *132.* article nr. 104207.

Benhabib, Jess, George W. Evans, and Seppo Honkapohja, "Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity?," *Journal of Economic Dynamics and Control*, 2014, *45*, 220–238.

——, **Stephanie Schmitt-Grohe, and Martin Uribe**, "The Perils of Taylor Rules," *Journal of Economic Theory*, 2001, *96*, 40–69.

Bernanke, B., "Monetary Policy in a New Era," Peterson Institute Conference on Rethinking Macroeconomic Policy, http:// https://piie.com/events/rethinking-macroeconomic-policy 2017.

Bernanke, B.S., M. Kiley, and J. Roberts, "Monetary Policy Strategies for a Low-Rate Environment," *AEA Papers and Proceedings*, 2019, *109*, 421–426.

Bhattarai, S. and C. Neely, "A Survey of the Empirical Literature on U.S. Unconventional Monetary," Working Paper, Federal Reserve Bank of St. Louis Working Paper 2016-021A 2016.

Bianchi, Francesco, Leonardo Melosi, and Matthias Rottner, "Hitting the Elusive Inflation Target," *Journal of Monetary Economics*, 2021, *124*, 107–122.

Branch, William, "Restricted Perceptions Equilibria and Learning in Macroeconomics." In Colander, ed (2006) pp. 135–160.

Branch, William A. and George W. Evans, "A Simple Recursive Forecasting Model," *Economic Letters*, 2006, *91*, 158–166.

Budianto, F., D. Nakata, and S. Schmidt, "Average Inflation Targeting and the Interest Rate Lower Bound," *European Economic Review*, 2023, *152.* article nr. 104384.

Bullard, James, "Speech: A Primer on Price Level Targeting in the U.S.," Slides 2018.

—— **and Kaushik Mitra**, "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 2002, *49*, 1105–1129.

Calvo, Guillermo A., "Staggered Pricing in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 1983, *12*, 383–398.

Carney, Mark, "Guidance," Speech, www.bankofcanada.ca 2012.

Cobham, David, Øyvind Eitrheim, Stefan Gerlach, and Jan F. Qvigstad, eds, *Twenty Years of Inflation Targeting: Lessons Learned and Future Prospects*, Cambridge: Cambridge University Press, 2010.

Cogley, Timothy and Thomas J. Sargent, "Anticipated Utility and Rational Expectations as Approximations of Bayesian Decision Making," *International Economic Review*, 2008, *49*, 185–221.

Coibion, O., Y. Gorodnichenko, E. Knotek II, and R. Schoenle, "Average Inflation Targeting and Household Expectations," Federal Reserve Bank of Cleveland, Working Paper No. 20-26 2020.

Colander, D, ed., *Post Walrasian Macroeconomics*, Cambridge, U.K.: Cambridge, 2006.

Dell'Ariccia, G., P. Rabanal, and D. Sandri, "Unconventional Monetary Policies in the Euro Area, Japan, and the United Kingdom," *Journal of Economic Perspectives*, 2018, *32*.

Elaydi, Saber, *An Introduction to Difference Equations*, New York, New York: Springer Science+Business Media Inc., 2005.

Eusepi, Stefano and Bruce Preston, "The Science of Monetary Policy: An Imperfect Knowledge Perspective," *Journal of Economic Literature*, 2018, *56*, 3–59.

___ **, Marc Giannoni, and Bruce Preston**, "Some Implications of Learning for Price Stability," *European Economic Review*, 2018, *106*, 1–20.

Evans, Charles L., "Monetary Policy in a Low-Inflation Environment: Developing a State-Contingent Price-Level Target," *Journal of Money, Credit and Banking*, 2012, *44*, 147–155.

Evans, George W. and Seppo Honkapohja, *Learning and Expectations in Macroeconomics*, Princeton, New Jersey: Princeton University Press, 2001.

___ **and** ___ , "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies*, 2003, *70*, 807–824.

___ **and** ___ , "Monetary Policy, Expectations and Commitment," *Scandinavian Journal of Economics*, 2006, *108*, 15–38.

___ **and** ___ , "Learning and Macroeconomics," *Annual Review of Economics*, 2009, *1*, 421–451.

___ **and** ___ , "Robust Learning Stability with Operational Monetary Policy Rules." In Schmidt-Hebbel and Walsh, eds (2009) chapter 5, pp. 145–170.

___ **and** ___ , "Expectations, Deflation Traps and Macroeconomic Policy." In Cobham, Eitrheim, Gerlach and Qvigstad, eds (2010) chapter 12, pp. 232–260.

___ **, Eran Guse, and Seppo Honkapohja**, "Liquidity Traps, Learning and Stagnation," *European Economic Review*, 2008, *52*, 1438–1463.

___ **, Seppo Honkapohja, and Kaushik Mitra**, "Does Ricardian Equivalence Hold When Expectations are not Rational?," *Journal of Money, Credit and Banking*, 2012, *44*, 1259–1283.

——, ——, and ——, "Expectations, Stagnation and Fiscal Policy: a Nonlinear Analysis," Discussion Paper 15171, CEPR 2020.

Gali, Jordi, *Monetary Policy, Inflation, and the Business Cycle*, Princeton New Jersey: Princeton University Press, 2008.

Gardner, R.B. and N.K. Govil, "Enestrom-Kakeya Theorem and Some of Its Generalizations." In Joshi, Dorff and Lahiri, eds (2014) pp. 1–33.

Hamilton, James D., *Time Series Analysis*, Princeton, NJ: Princeton University Press, 1994.

Hebden, J., E. Herbst, J. Tang, G. Topa, and F. Winkler, "How Robust Are Makeup Strategies to Key Alternative Assumptions?," Finance and Economics Discussion Series 2020-069. Washington: Board of Governors of the Federal Reserve System, https://doi.org/10.17016/FEDS.2020.069 2020.

Hoffman, Mattias, Emanuel Moench, Lora Pavlova, and Guido Schultefrankenfeld, "Would Households Understand Average Inflation Targeting," *Journal of Monetary Economics, Supplement*, 2022, *129*, S52–S66.

Hommes, Cars H. and Mei Zhu, "Behavioral Learning equilibria," *Journal of Economic Theory*, 2014, *150*, 778–814.

Honkapohja, Seppo and Kaushik Mitra, "Price Level Targeting with Evolving Credibility," *Journal of Monetary Economics*, 2020, *116*, 88–103, https://doi.org/10.1016/j.jmoneco.2019.09.009.

IMF, *World Economic Outlook, October 2022*, Washington, D.C.: IMF Publication Services, 2022.

Jacobs, D.P., E. Kalai, and M. Kamien, eds, *Frontiers of Research in Economic Theory*, Cambridge: Cambridge University Press, 1998.

Jia, Chengcheng and Jing Cynthia Wu, "Average Inflation Targeting: Time Inconsistency and Intentional Ambiguity," Working Paper 2022.

Joshi, S., M. Dorff, and I. Lahiri, eds, *Current Topics in Pure and Computational Complex Analysis*, Springer-India, 2014.

Jury, E.I., "A Simplified Stability Criterion for Linear Discrete Systems," *Proceedings of the IRE*, 1961, *49*, 1493–1500.

Kreps, David M., "Anticipated Utility and Dynamic Choice." In Jacobs, Kalai and Kamien, eds (1998) pp. 242–274.

Kuttner, K., "Outside the Box: Unconventional Monetary Policy in the Great Recession and Beyond," *Journal of Economic Perspectives*, 2018, *32.*

Leeper, Eric M., "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of Monetary Economics*, 1991, *27*, 129–147.

Mertens, T. M. and J.C. Williams, “Tying Down the Anchor: Monetary Policy Rules and the Lower Bound on Interest Rates,” Staff Report No.887, Federal Reserve Bank of New York 2019.

Milani, Fabio, “Expectations, Learning and Macroeconomic Persistence,” *Journal of Monetary Economics*, 2007, *54*, 2065–2082.

Moessner, R., D-J. Jansen, and J. de Haan, “Communication about Future Policy Rates in Theory and Practice: a Survey,” *Journal of Economic Surveys*, 2017, *31.*

Nessen, Marianne and David Vestin, “Average Inflation Targeting,” *Journal of Money, Credit and Banking*, 2005, *37*, 837–863.

Orphanides, Athanasios and John C. Williams, “Inflation Scares and Forecast-Based Monetary Policy,” *Review of Economic Dynamics*, 2005, *8*, 498–527.

Powell, Jerome H., “New Economic Challenges and the Fed Monetary Policy Review,” Remarks 2020.

Reifschneider, David and John C. Williams, “Three Lessons for Monetary Policy in a Low-Inflation Era,” *Journal of Money, Credit and Banking*, 2000, *32*, 936–966.

Rotemberg, Julio J., “Sticky Prices in the United States,” *Journal of Political Economy*, 1982, *90*, 1187–1211.

Salle, Isabelle, “What To Target? Insights from a lab experiment *,” Bank of Canada Working Paper forthcoming 2021.

Sargent, Thomas J., *The Conquest of American Inflation*, Princeton NJ: Princeton University Press, 1999.

____ , “Evolution and Intelligent Design,” *American Economic Review*, 2008, *98*, 5–37.

Schmidt-Hebbel, Klaus and Carl E. Walsh, eds, *Monetary Policy under Uncertainty and Learning*, Santiago: Central Bank of Chile, 2009.

Svensson, Lars E.O., “Monetary Policy Strategies for the Federal Reserve,” *International Journal of Central Banking*, 2020, *16*, 133–193.

Williams, John C., “Preparing for the Next Storm: Reassessing Frameworks and Strategies in a Low R-star World,” *FRB Economic Letter 2017-13*, 2017, pp. 1–8.

Figures

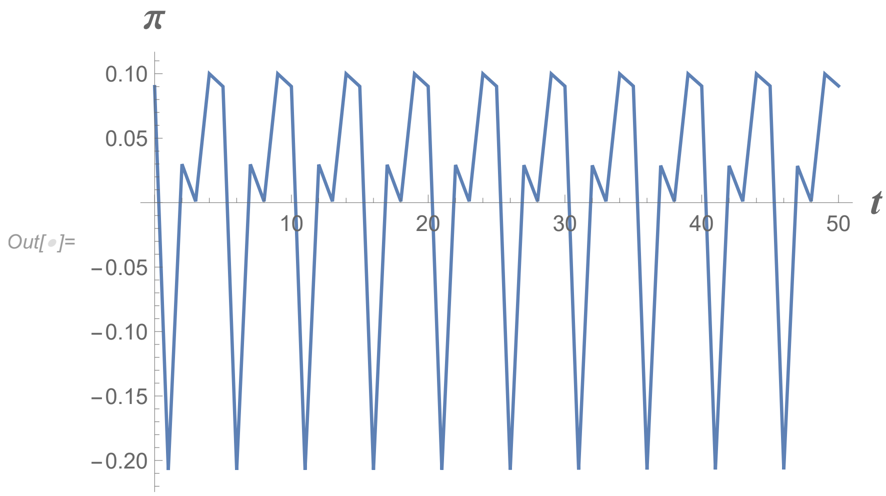

Figure I: Unstable dynamics with AIT in Fisher model

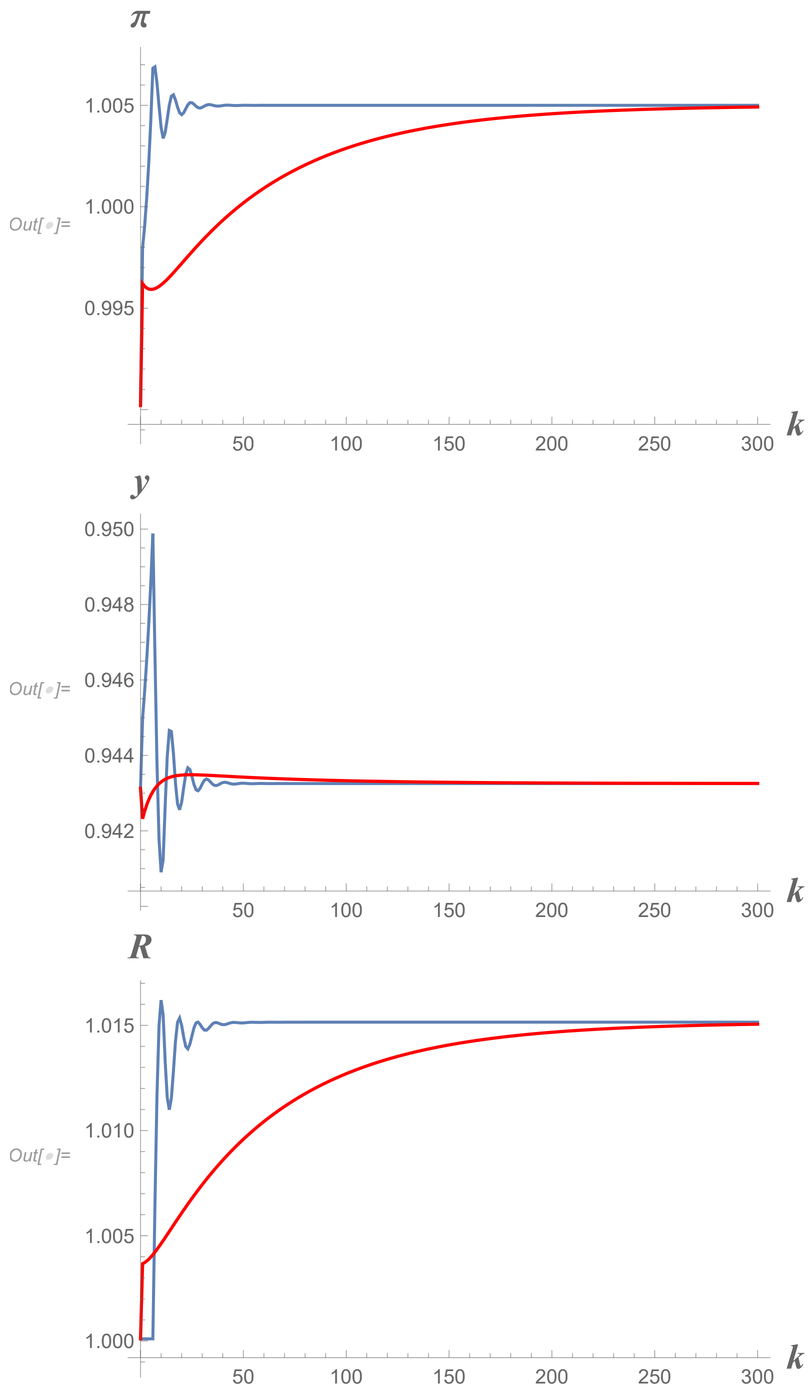

Figure II: Escape of inflation, output and interest rate from liquidity trap under AIT with gs (blue) and IT (red)

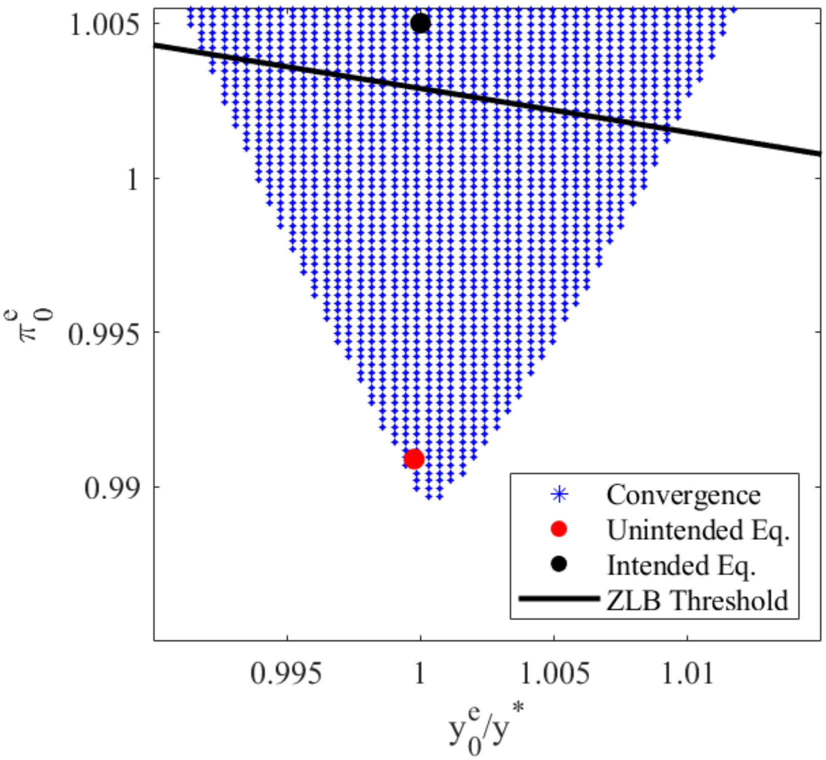

Figure III: Domain of escape to target steady state

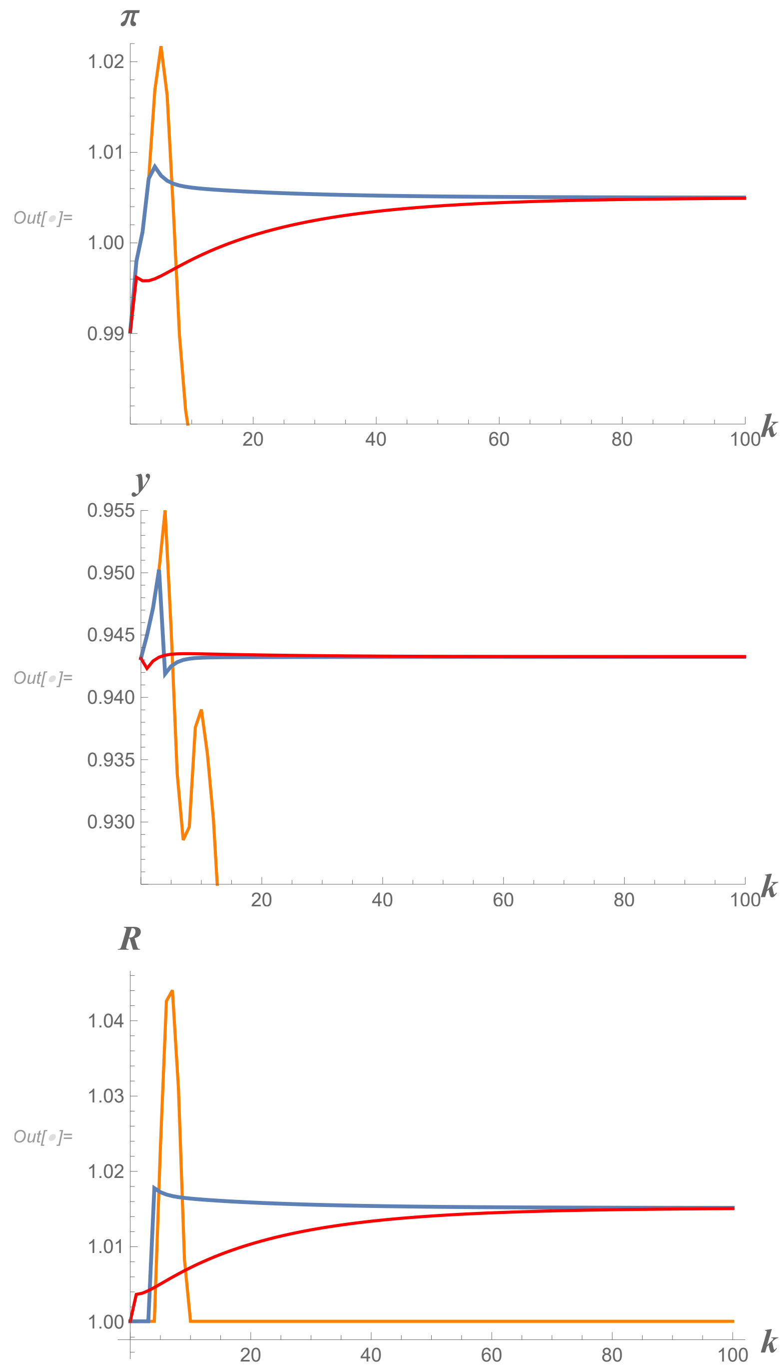

Figure IV: Escape of inflation, output and interest rate from liquidity trap under asymmetric AIT (blue), IT (red) and symmetric AIT (yellow)

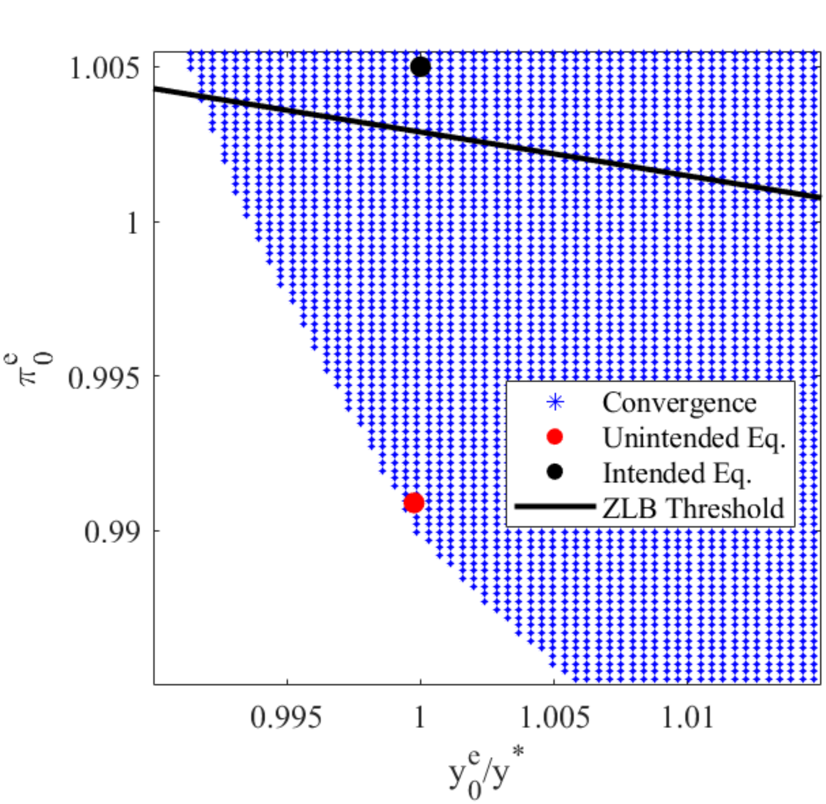

Figure A.1: Domain of escape for IT

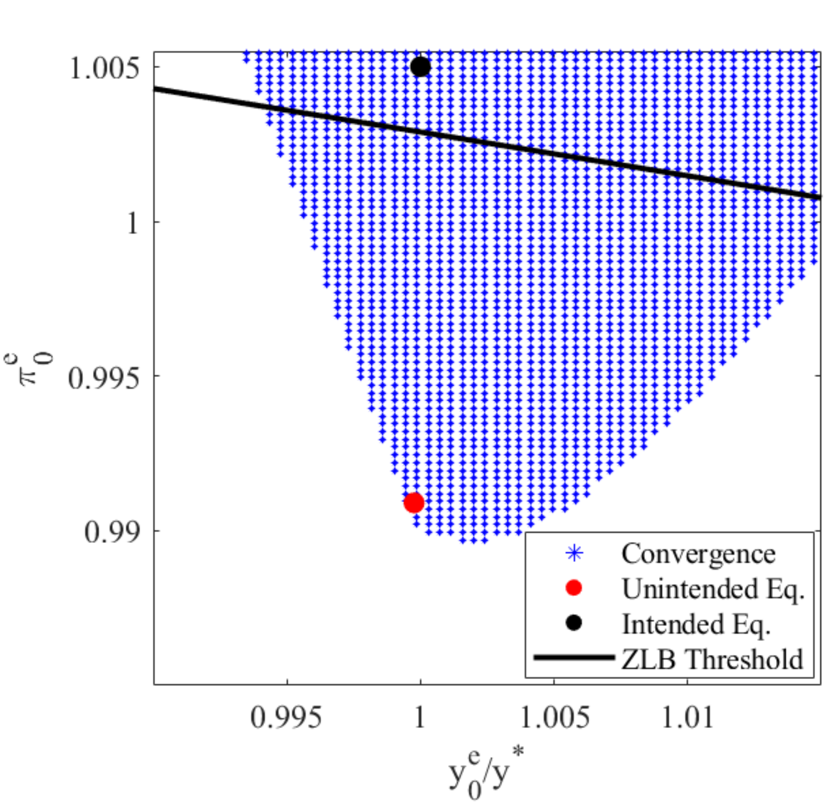

Figure A.2: Domain of escape for AIT with MSV Beliefs